A complete characterization of group-strategyproof mechanisms of cost-sharing

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[1] joint work with Angelina Vidali

Model

- A set of agents $\mathcal{A}$ interested in the service
- Each agent i has a private value for the service v_i
- **Mechanism:** Elicts a bid b_i from each agent and decides
  1. the set of serviced players $O(b)$
  2. the payment of each player $p_i(b)$
- **Utility:** $v_i \cdot a_i(b) - p_i(b)$
  $a_i(b)$: binary indicator for $i \in O(b)$

Axioms of Cost-sharing

- **Non Positive Transfer**
  The payments are non-negative
- **Voluntary Participation**
  Only the serviced players may be charged and not greater than their bids

  Negative bid: no service
- **Consumer Sovereignty**
  Guaranteed to receive service if announced a high enough bid $b_i^* \in \mathbb{R}$

Group-strategyproofness

Successful Coalition: $S \subseteq \mathcal{A}$

- The players in $\mathcal{A} \setminus S$ report their true values
- Compared with the truthful scenario:
  - The utility of every $i \in S$ does not decrease
  - The utility of at least one $i \in S$ strictly increases

Group-strategyproof mechanism: No successful coalitions

Budget Balance

- $C(S)$ cost of servicing the set S
- α-**Budget balance**: $\alpha \cdot C(O(b)) \leq \sum_{i \in \mathcal{A}} p_i(b) \leq C(O(b))$
- No assumption about budget balance

Cost-sharing Schemes

- A cost sharing scheme is a function $\xi : A \times 2^A \to \mathbb{R}^+ \cup \{0\}$, where $i \notin S \Rightarrow \xi(i, S) = 0$.
- (Immorlica et. al. 05) The payment function of a group-strategyproof cost-sharing mechanism corresponds to a cost-sharing scheme
- **Main prolbem**: Characterize the cost-sharing schemes that give rise to group-strategyproof mechanisms (along with the other properties)

Cross Monotonicity

- **Cross Monotonicity**: For all $S, T \subseteq \mathcal{A}$ and $i \in S$: $\xi(i, S) \geq \xi(i, S \cup T)$.
- Sufficient property for group-strategyproofness (Moulin 99)
- Not necessary and also poor budget balance for many important combinatorial problems (Immorlica et. al. 05)

Semi-cross Monotonicity

- **Semi-cross Monotonicity**: For all $S \subseteq \mathcal{A}$ and $i \in S$, either

  $\forall j \in S \setminus \{i\}$: $\xi(j, S \setminus \{i\}) \geq \xi(j, S)$ or
  $\forall j \in S \setminus \{i\}$: $\xi(j, S \setminus \{i\}) \leq \xi(j, S)$.

- Necessary property for group-strategyproofness, however not sufficient (Immorlica et. al. 05)

Example

ξ	1	2	3
$\{1,2,3\}$	20	10	10
$\{1,2\}$	20	10	–
$\{1,3\}$	20	–	20
$\{1\}$	10	–	–

b_1	b_2	b_3	$O(b)$

Example

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b_1	b_2	b_3	$O(b)$
b_1^*	10	15	$\{1,2,3\}$
b_1^*	-1	15	$\{1\}$

Example

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$\{1,2\}$	20	10	–
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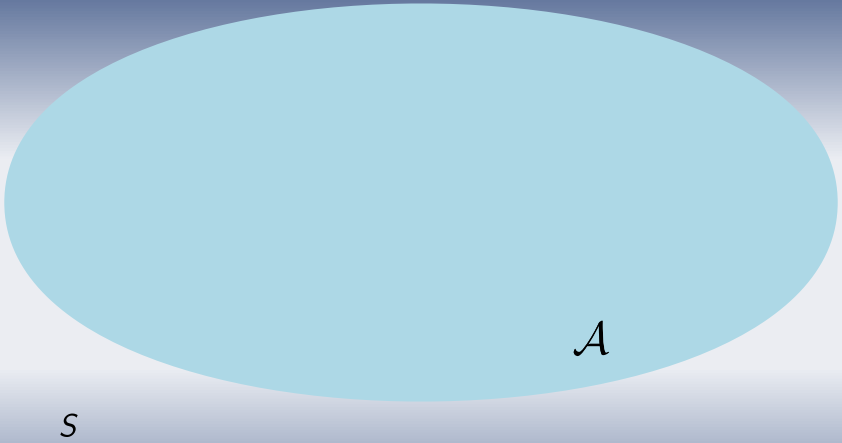
b_1	b_2	b_3	$O(b)$
b_1^*	10	15	$\neq \{1,2,3\}$
b_1^*	b_2^*	b_3^*	$\{1,2,3\}$

Our Characterization



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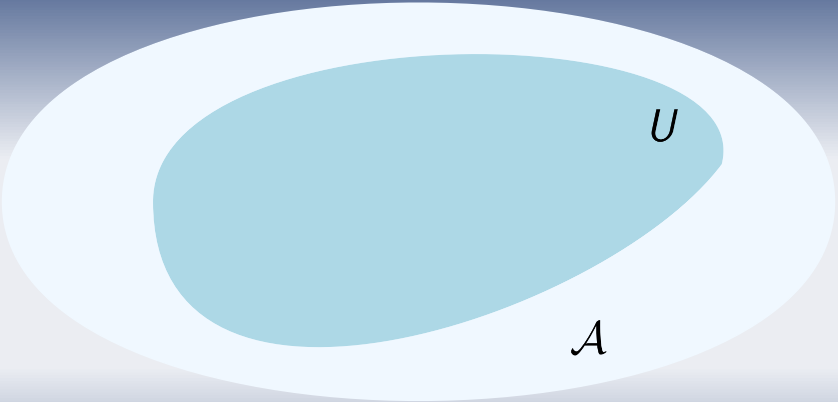
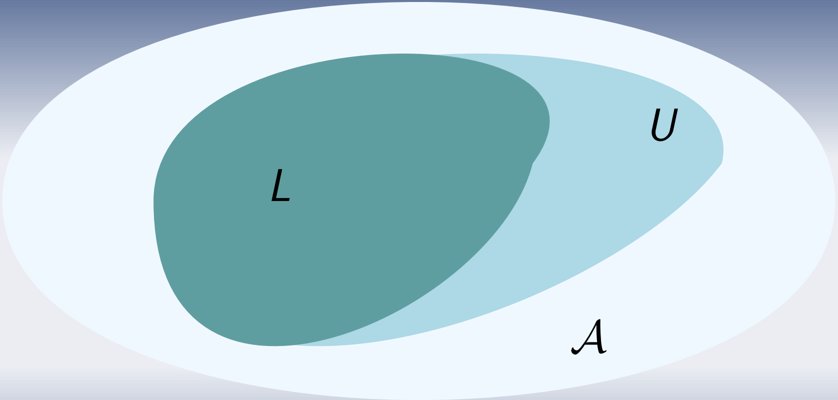


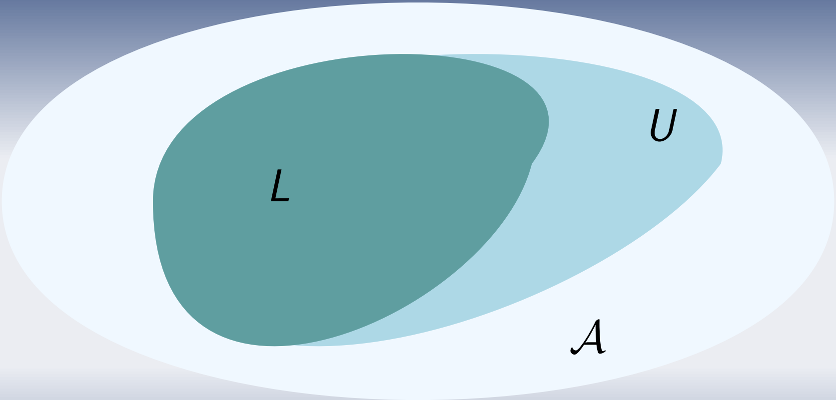
$S \subseteq U$

Our Characterization



$L \subseteq S \subseteq U$

Our Characterization



$$L \subseteq S \subseteq U : \qquad \xi^*(i, L, U) = \min \xi(i, S)$$
$$(i \in S)$$

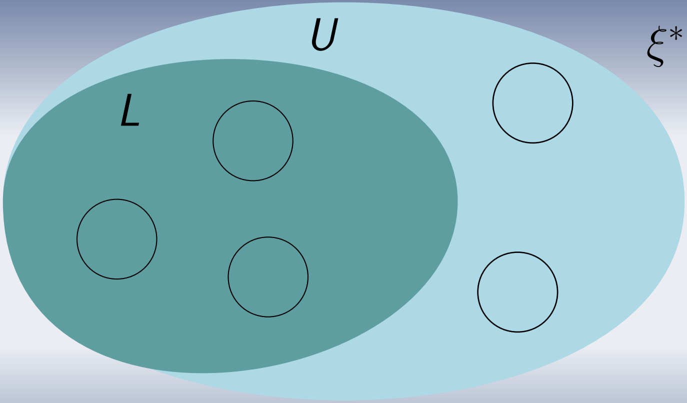
Our Characterization

- **Fence monotonicity**: a property of the cost-sharing schemes
- Consider all possible combinations for L and U
- Three properties should be satisfied

Fence Monotonicity: First property

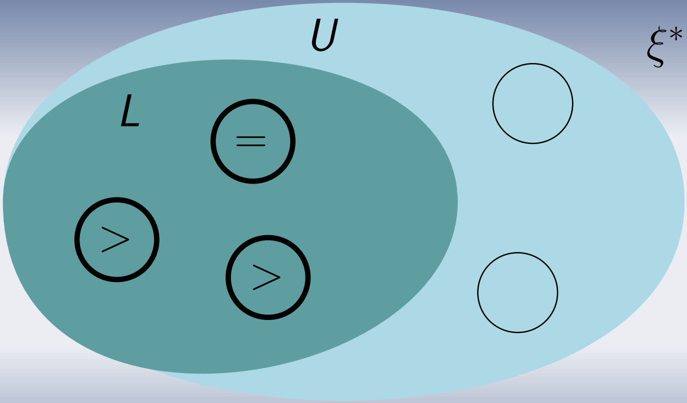
- Fix any pair L, U: Set S is optimal if every $i \in S$ is charged $\xi^*(i, L, U)$
- **First Condition**: There is at least one optimal set
- Equivalent with Semi-cross-monotonicity for $|U \setminus L| = 1$

Fence Monotonicity: First property



$\xi^*(i, L, U)$

Fence Monotonicity: First property



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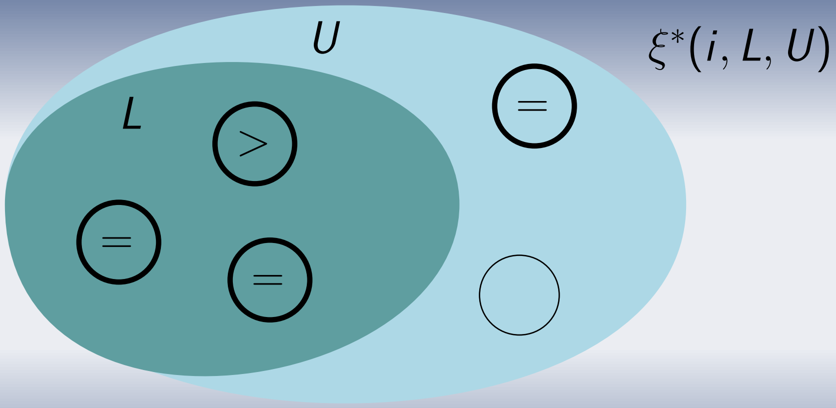

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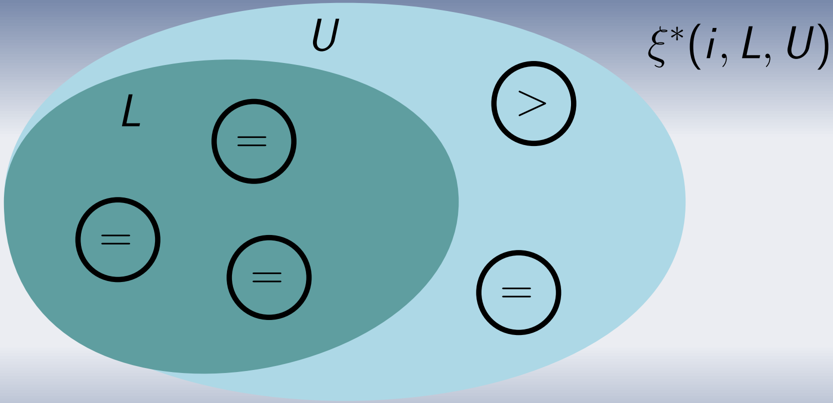

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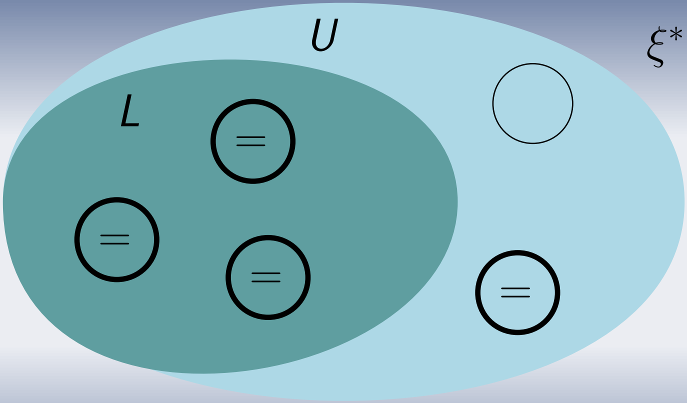
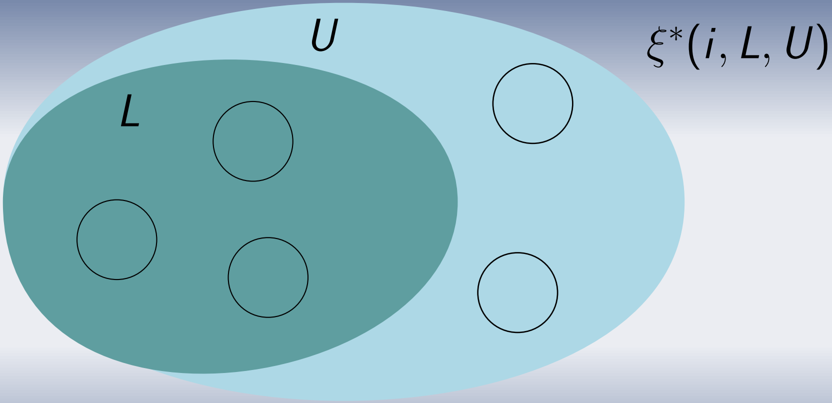


$\xi^*(i, L, U)$

Fence Monotonicity: Second property

- Fix any pair L, U: Set S is weakly optimal if at least every $i \in S \setminus L$ is charged $\xi^*(i, L, U)$
- **Second Condition**: Every $i \in U \setminus L$ belongs to a weakly optimal set
- Every $i \in U \setminus L$ belongs to an optimal set $\Leftrightarrow$ Cross monotonicity

Fence Monotonicity: Second property



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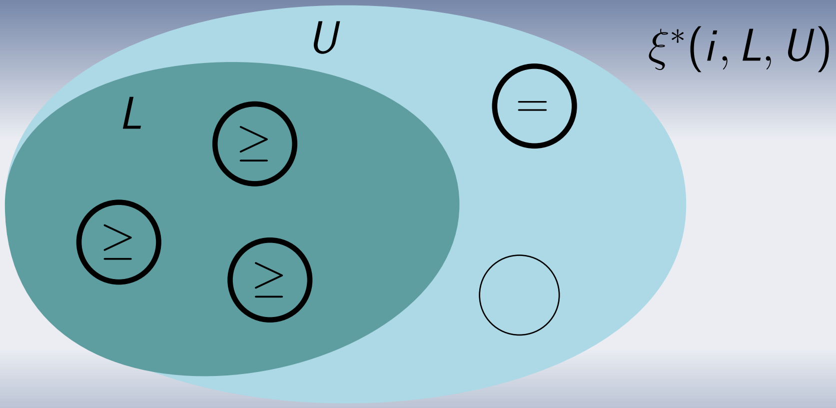

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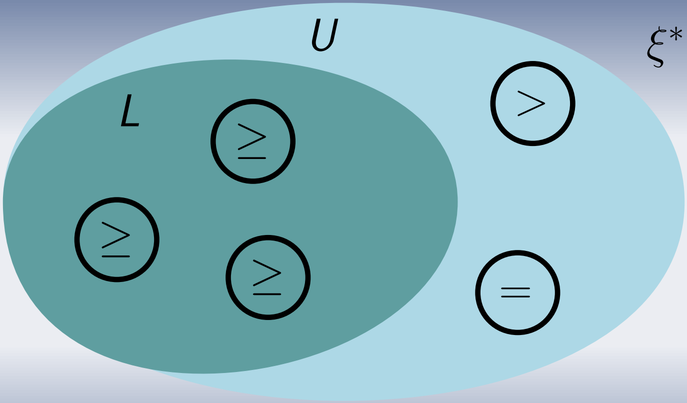
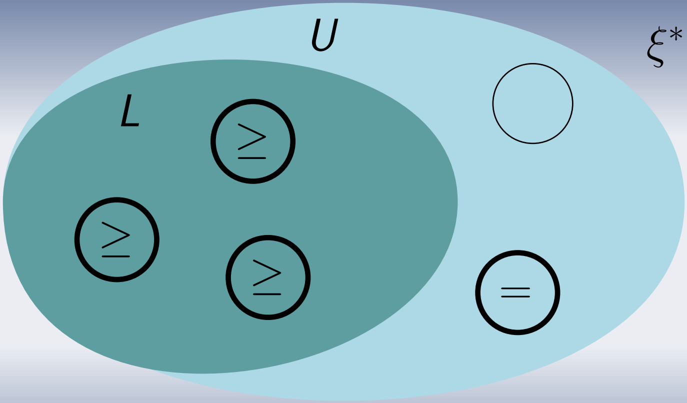


$\xi^*(i, L, U)$

Fence Monotonicity: Second property

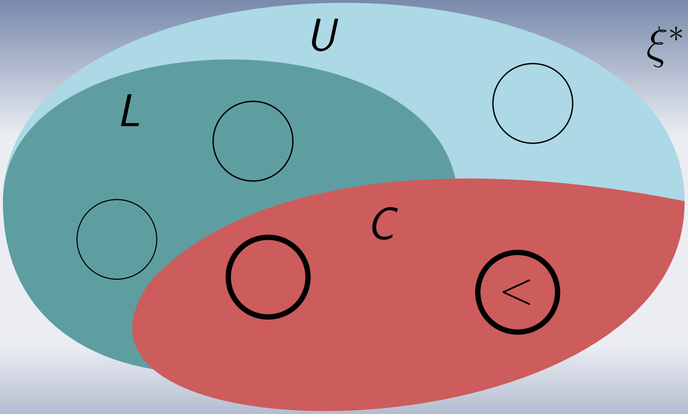


$\xi^*(i, L, U)$

Fence Monotonicity: Third property

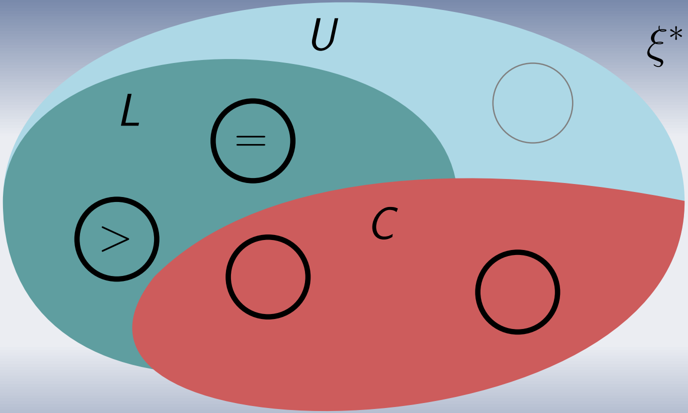
- Fix any pair L, U and consider any $C \subset U$ where at least one $j \in C$ is charged less than $\xi^*(j, L, U)$ ($L \not\subseteq C$)
- **Third Property:** There exists one set T
  1. non-empty $T \subseteq L \setminus C$
  2. every $i \in T$ is charged $\xi^*(i, L, U)$ at $C \cup T$

Fence Monotonicity: Third property



$\xi^*(i, L, U)$

Fence Monotonicity: Third property



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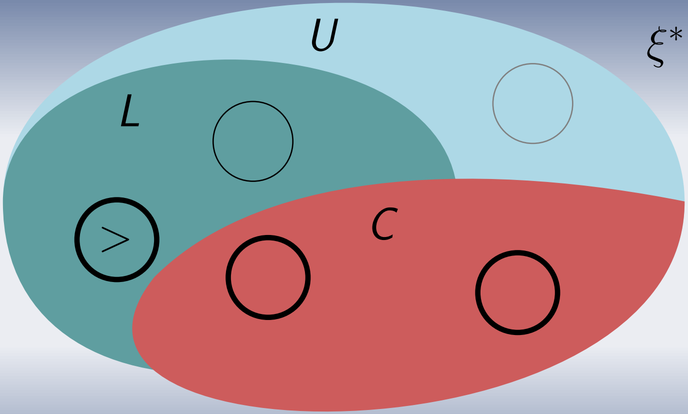
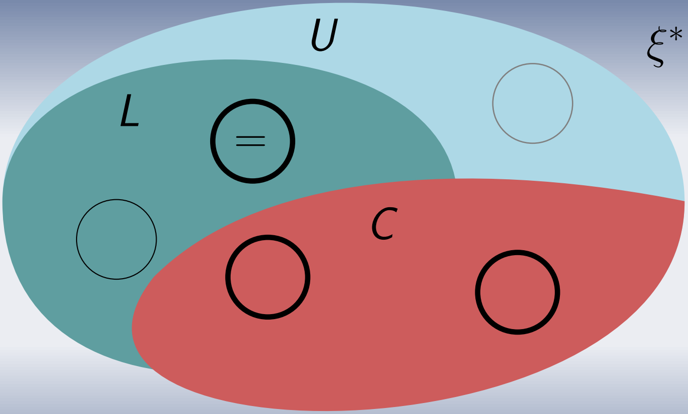
Fence Monotonicity: Third property



$\xi^*(i, L, U)$

Fence Monotonicity: Third property



$\xi^*(i, L, U)$

Theorem

Theorem

The cost sharing scheme of any group-strategyproof mechanism satisfies Fence Monotonicity.

Fencing mechanisms

- Pair L, U is **stable** at b, iff
  1. $\forall i \in L,\ b_i > \xi^*(i, L, U)$
  2. $\forall i \in U \setminus L,\ b_i = \xi^*(i, L, U)$
  3. $\forall R \subseteq \mathcal{A} \setminus U$ with $R \neq \emptyset$, $\exists i \in R$: $b_i < \xi^*(i, L, U \cup R)$
- Given any pair L, U and any bid vector b, a tie-breaking rule $\sigma(L, U, b) = S \subseteq \mathcal{A}$ is valid if S is optimal w.r.t. L, U

Fencing mechanisms

Algorithm 1 Fencing mechanism

Require: Fence monotone ξ, valid tie-breaking rule σ for ξ, and bid vector b

Find stable pair L, U

$S \leftarrow \sigma(L, U, b)$

return $O(b) \leftarrow S$ and for all $i \in \mathcal{A}$, $p_i(b) \leftarrow \xi(i, S)$

Theorem

Theorem

A mechanism is group-strategyproof if and only if it is a Fencing Mechanism.

Budget Balance and Complexity

Theorem

There is no general group-strategyproof mechanism with constant budget balance.

Theorem

Finding the stable pair of an input is no harder than computing the outcome of a group-strategyproof mechanism given polynomial access to ξ^**.*

Open Problems

1. **Budget Balance**:
   - Upper bounds for important combinatorial problems
   - Construct group-strategyproof mechanisms with better performance
2. **Complexity**
   - Find the complexity of computing the stable pair
   - Characterize tractable group-strategyproof mechanisms
3. **Others Characterizations**
   - Specific cost sharing problems (budget balance restrictions)
   - The weaker version of weakly group-strategyproof mechanisms (Mehta et. al.)
   - Group-strategyproof mechanisms in other domains

THANK YOU!