Scheduling with Precedence Constraints of Low Fractional Dimension

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joint work with

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Manno, Switzerland

Approximation Algorithms

Many practical problems are unlikely to be solvable *efficiently*.
The necessity to solve them *somehow* leads to...

Approximation Algorithm (for Minimization Problem)

A ρ-approximation algorithm ($\rho \geq 1$) satisfies:

- it runs in polynomial time
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E.g: 2-approximation for Vertex Cover: covers the graph with at most twice as many vertices as necessary.

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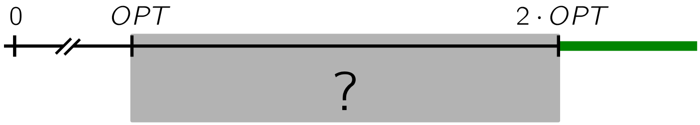

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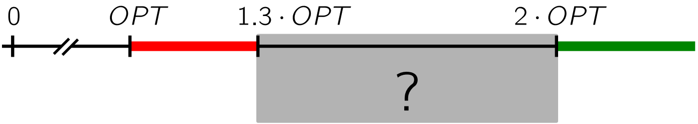

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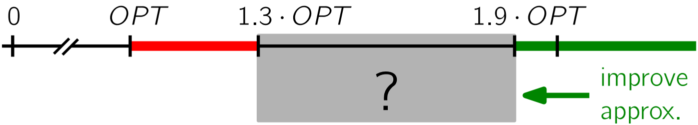

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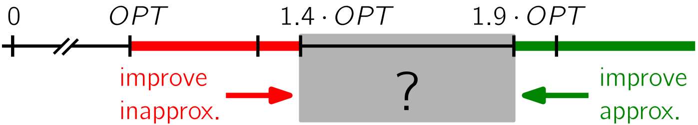

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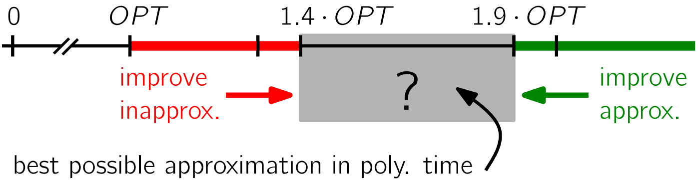

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Find: An ordering of the jobs, such that $\sum_i w_i C_i$ is minimized.

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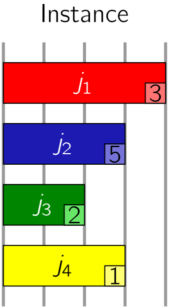

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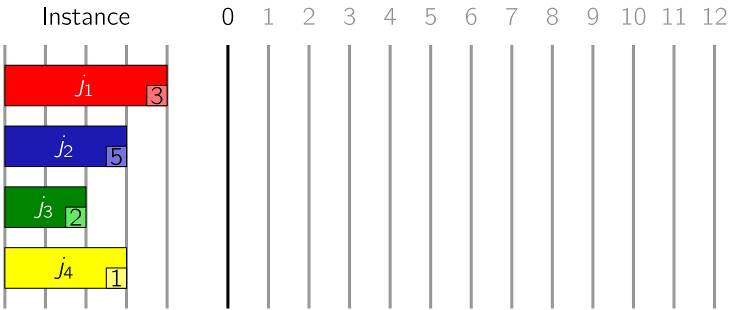

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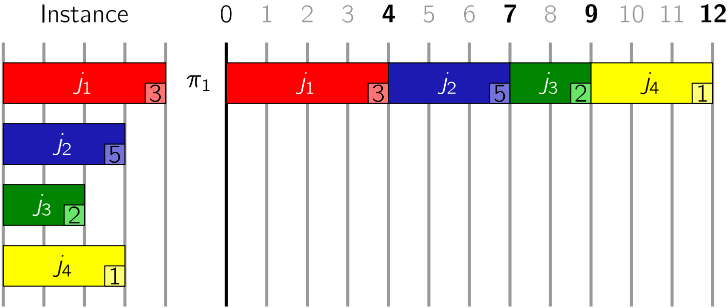

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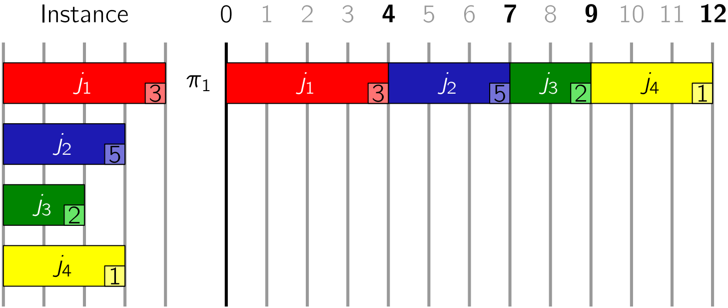

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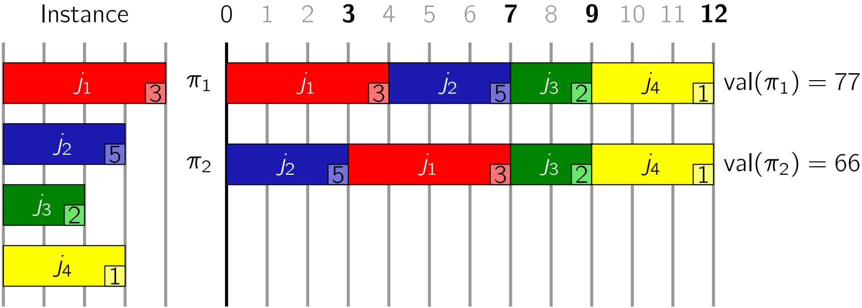

$$\text{val}(\pi_2) = 5 \cdot 3 + 3 \cdot 7 + 2 \cdot 9 + 1 \cdot 12 = 66$$

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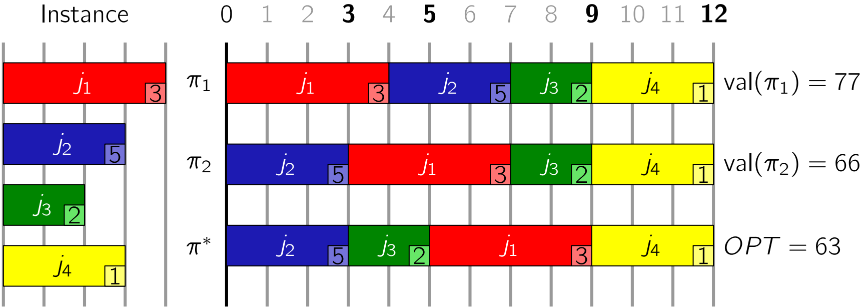

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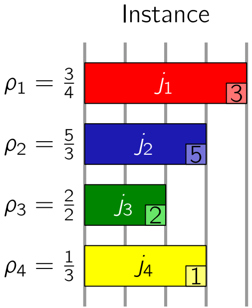
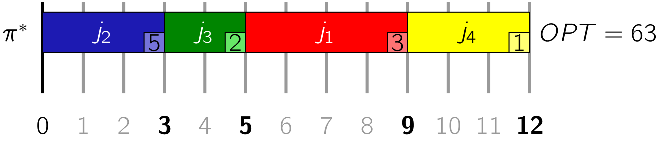


Theorem [SMITH'56]

Ordering non-increasingly according to $\rho := w/p$ is optimal.



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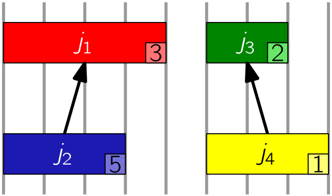

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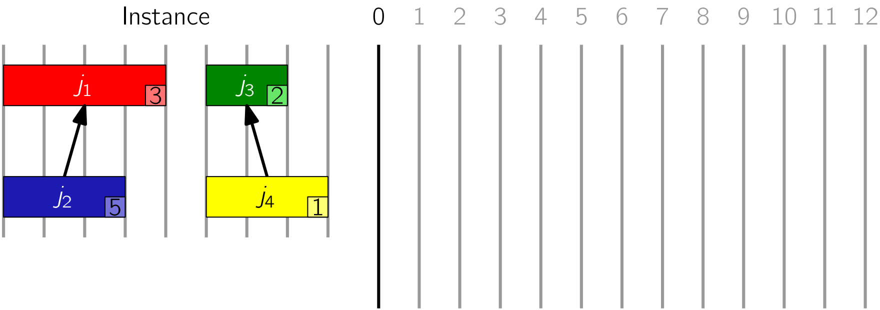

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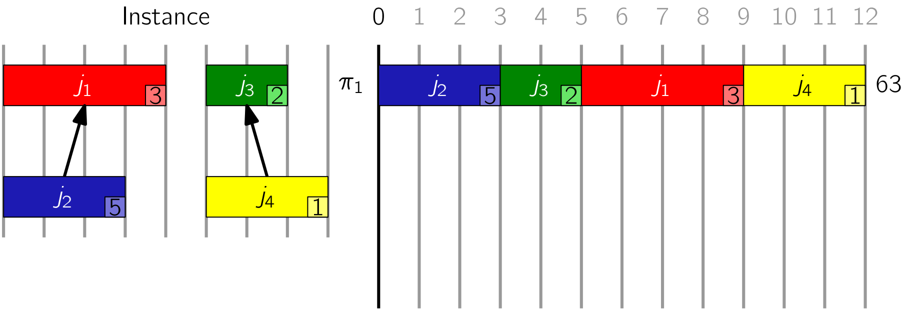

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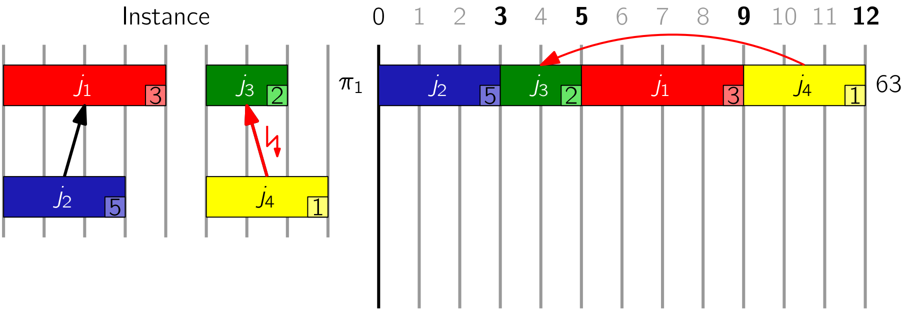

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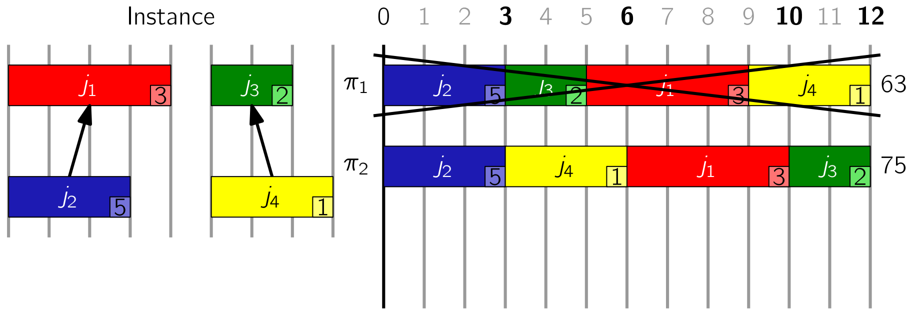

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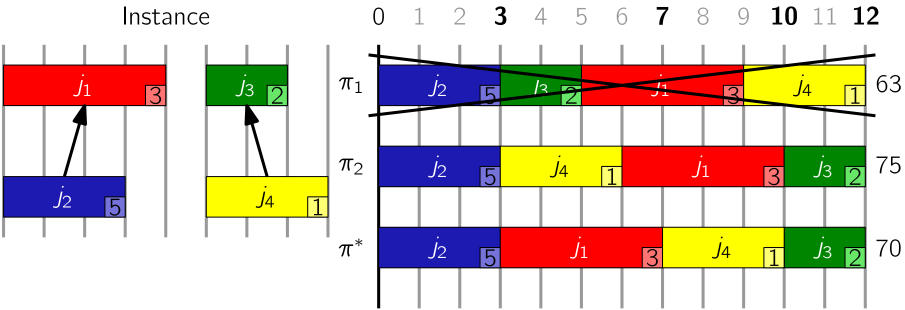

History & Literature

1. Extensively studied:
   - The general version is strongly NP-hard
     [Lawler'78], [Lenstra & Rinnooy-Kan'78]
   - Several 2-approximation algorithms
     [Schulz'96], [Hall,Schulz, Shmoys & Wein'97], [Chudak & Hochbaum'97], [Chekuri & Motwani'99], [Margot, Queyranne & Wang'03], [Pisaruk'03]
   - Better than 2-approximation for special precedence constraints
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   - No PTAS. Variable part as hard as vertex cover
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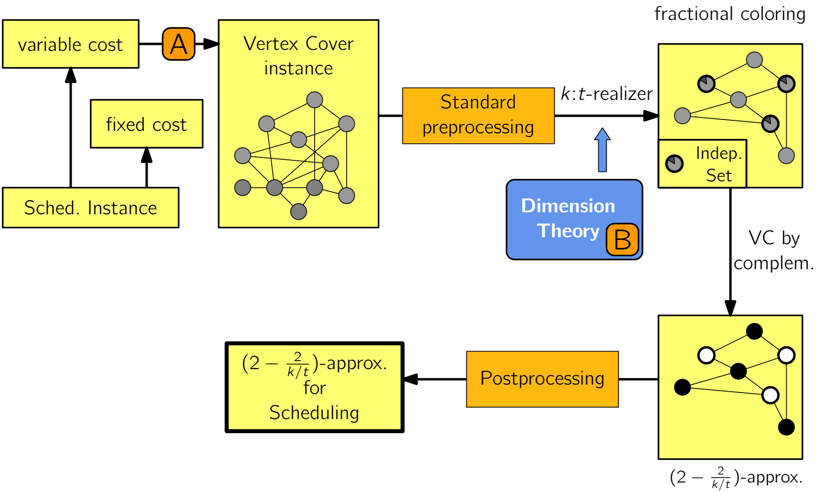
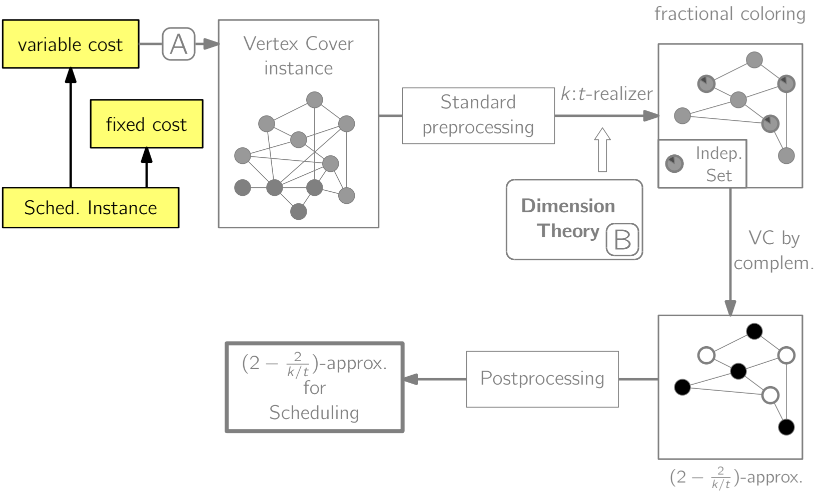
General framework

Yields a $(2 - 2/f)$-approximation algorithm whenever the fractional dimension of the poset is $\leq f$. (interval orders, bounded degree, ...)

The algorithmic framework - Overview



The algorithmic framework - Fixed cost



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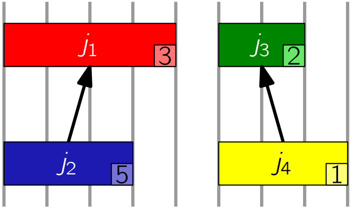

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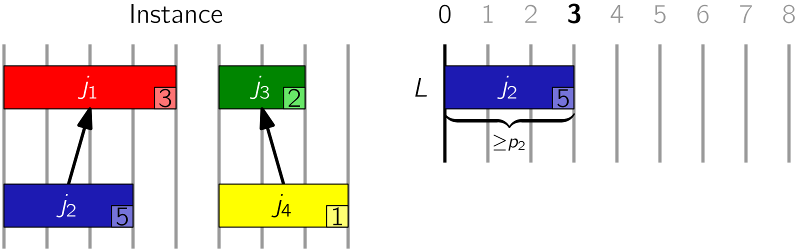

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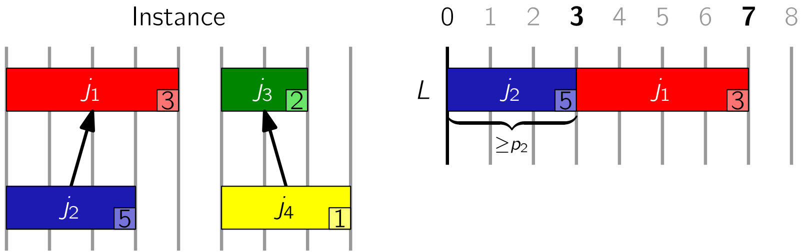

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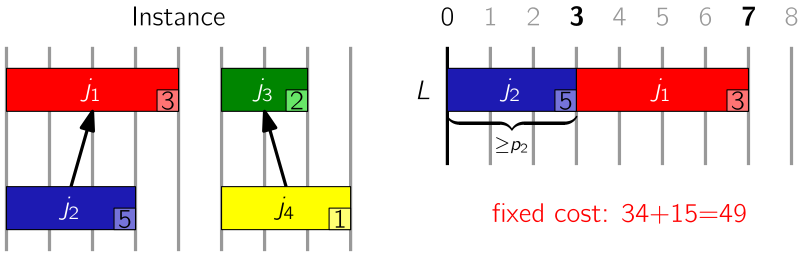
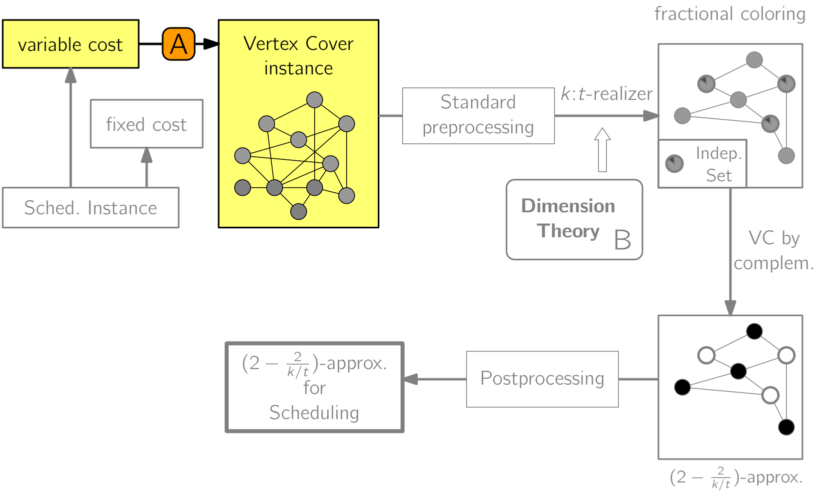


The algorithmic framework - Sched. & Vertex Cover



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Problem $1|prec|\sum_j w_j C_j$ *is a special case of* MINIMUM WEIGHTED VERTEX COVER.

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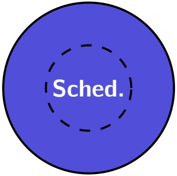

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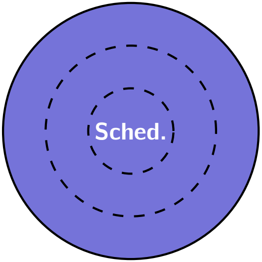

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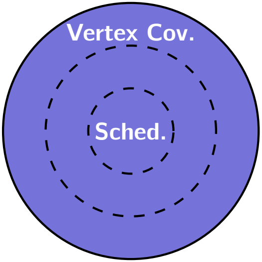

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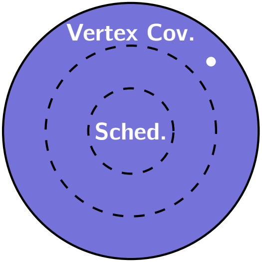

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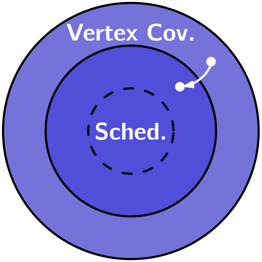

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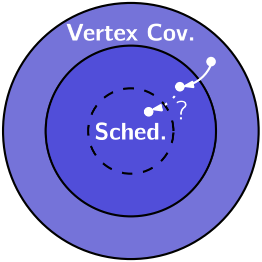

[Potts'80]

[Chudak & Hochbaum'99]

[Correa & Schulz'04]

The algorithmic framework - Sched. & Vertex Cover

Theorem

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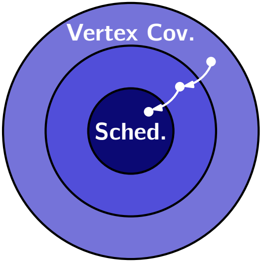


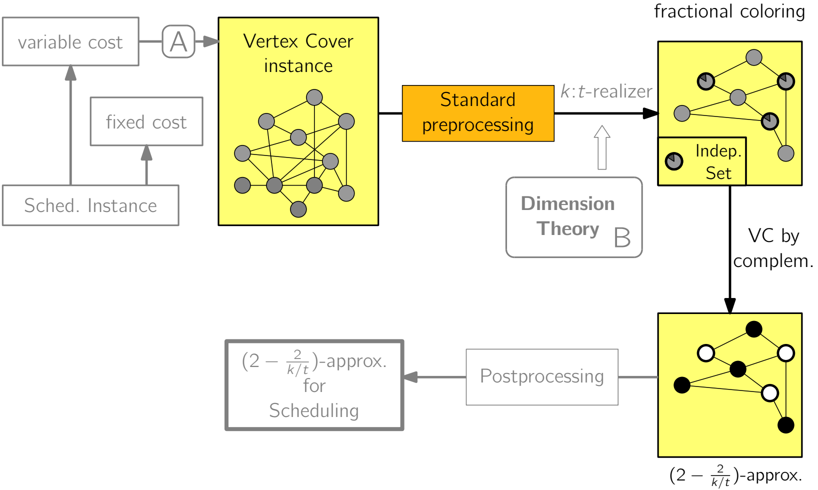
[Potts'80]

[Chudak & Hochbaum'99]

[Correa & Schulz'04]

[Ambühl & Mastrolilli'06]

The algorithmic framework - VC & coloring



The algorithmic framework - VC & coloring

Given: Graph $G(V, E)$.
Find: Coloring of V s.t. no $e \in E$ is "monochromatic".

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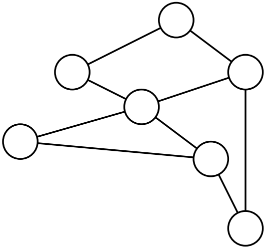

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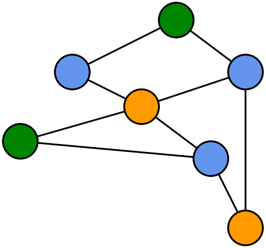

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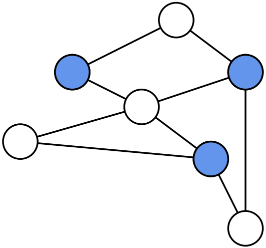

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The **heaviest color** weighs at least $\frac{W}{k}$ (average weight).

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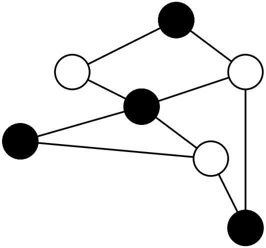

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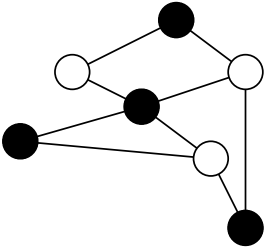

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The algorithmic framework - VC & *fractional* coloring

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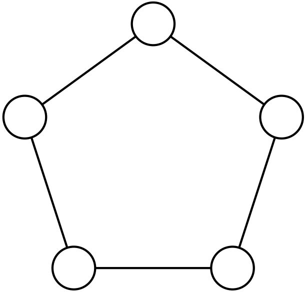

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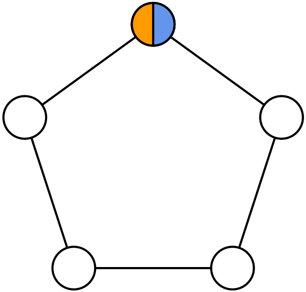

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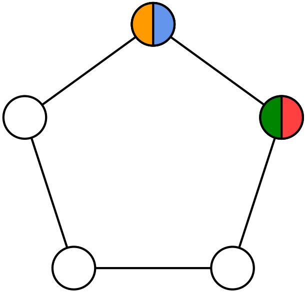

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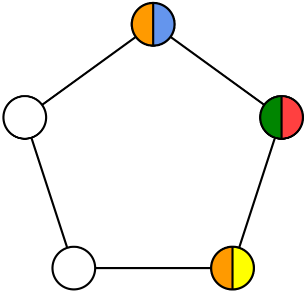

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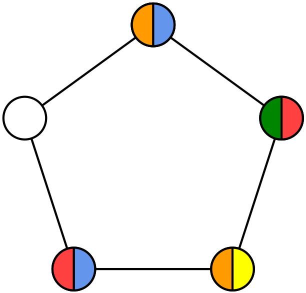

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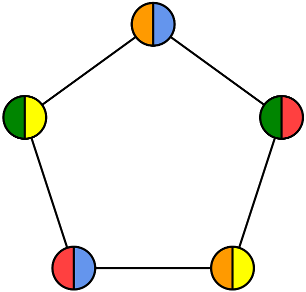

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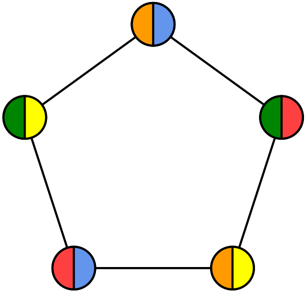

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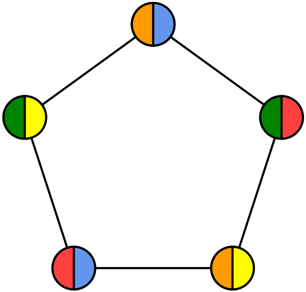

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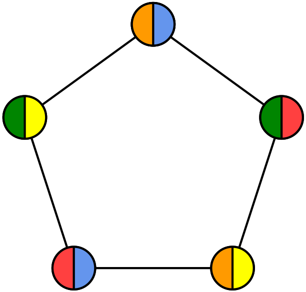

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Here: $\frac{6}{5}$

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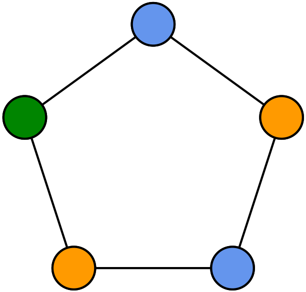

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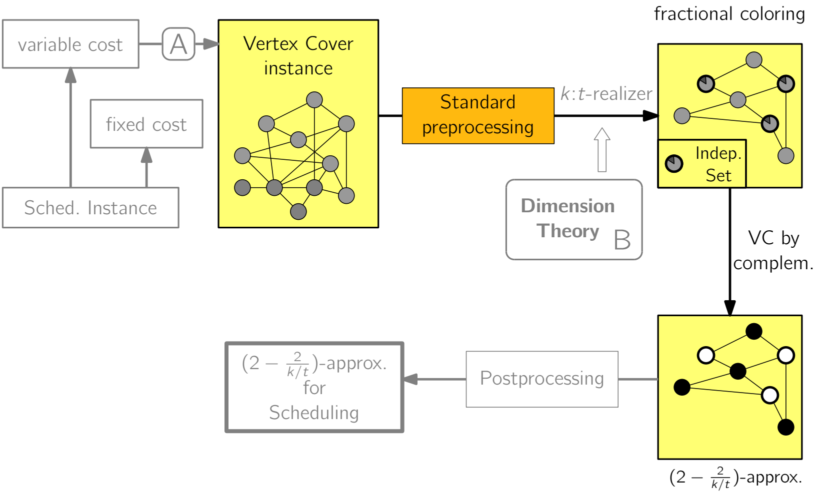
Using coloring [HOCHBAUM'83]:

$$\left(2 - \frac{2}{3}\right) = \frac{4}{3}$$

The algorithmic framework - VC & *fractional* coloring



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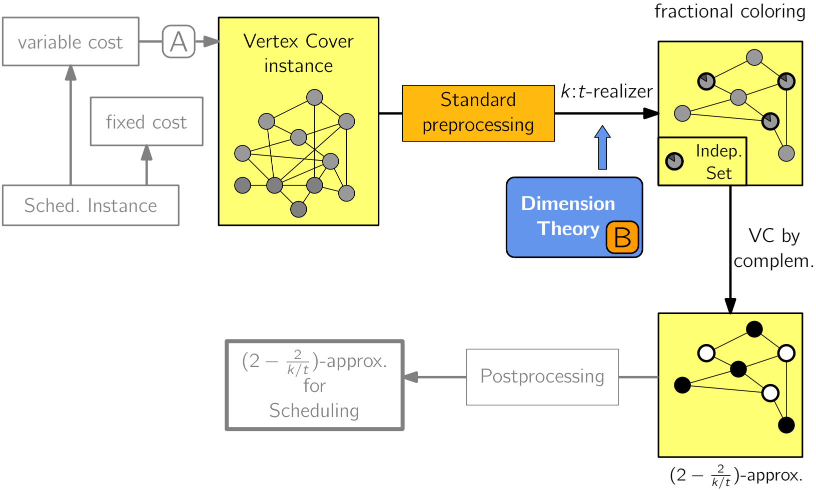

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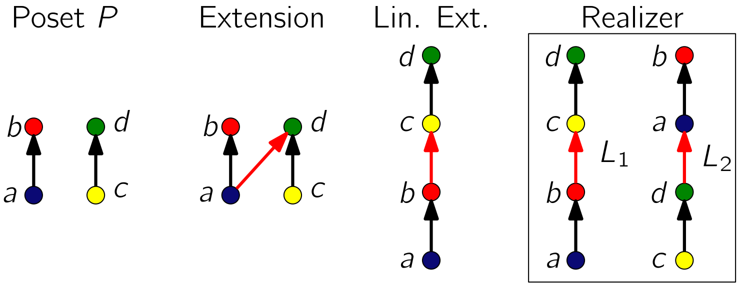
The algorithmic framework - Dimension Theory



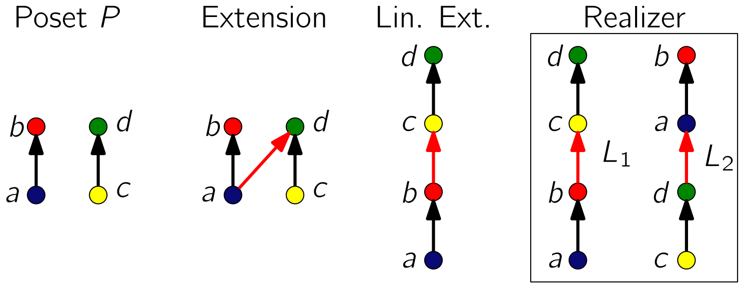
Definition

A *t*-realizer of a poset **P** is a set of *t* linear extensions of **P** s.t. any (ordered) incomparable pair is reversed in at least 1 linear extension .

Definition [DUSHNIK & MILLER, 1941]

The dimension of a poset **P** is the smallest *t* such that there exists a *t*-realizer of *P*.

The algorithmic framework - Dimension Theory



Definition

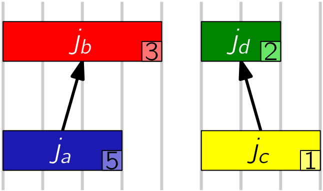
A *k*:*t*-realizer of a poset **P** is a set of *t* linear extensions of **P** s.t. any (ordered) incomparable pair is reversed in at least *k* linear extensions.

Definition [Brightwell & Scheinerman'92]

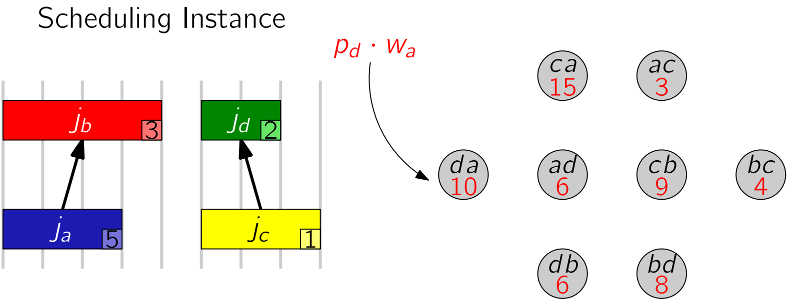
The fractional dimension of a poset **P** is the smallest t/k such that there exists a *k*:*t*-realizer of *P*.

The algorithmic framework - Graph Structure

Scheduling Instance

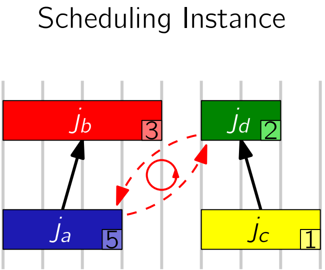


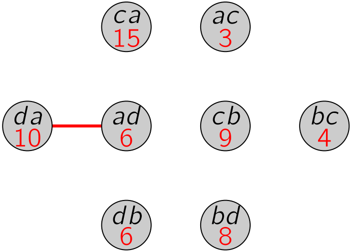
The algorithmic framework - Graph Structure



- A vertex for each (ordered) pair of incomparable jobs.
- Intuitively: If $(a, d) \in VC$ then job a is scheduled before job d.

The algorithmic framework - Graph Structure





- Edge type (1): Either schedule a before d or d before a.

The algorithmic framework - Graph Structure

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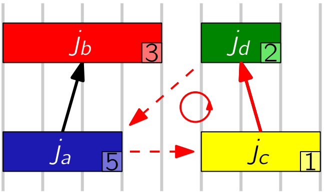
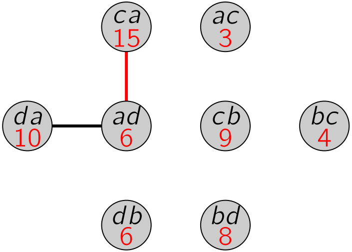
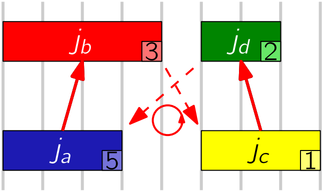


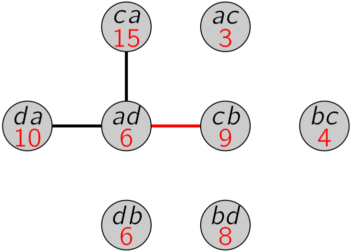


- Edge type (2): Schedule *c* before *a* or *a* before *d* to avoid cycles of this type.

The algorithmic framework - Graph Structure

Scheduling Instance

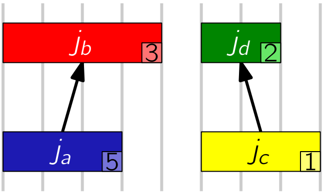


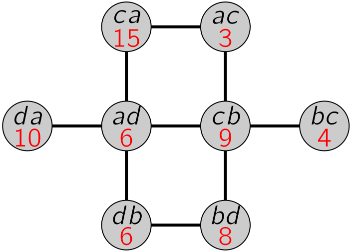


- Edge type (3): Schedule c before b or a before d to avoid cycles of this type.

The algorithmic framework - Graph Structure

Scheduling Instance





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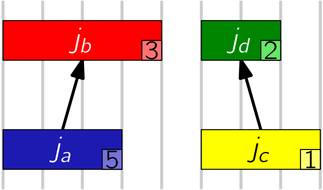
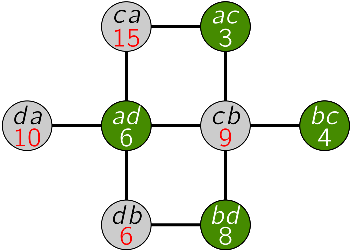
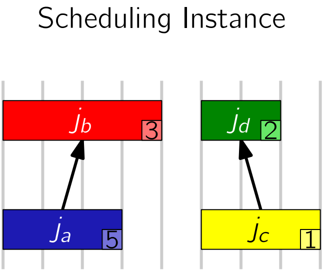


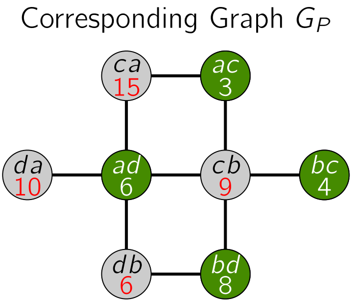
Corresponding Graph G_P



- The green nodes represent an optimal vertex cover with value $3 + 6 + 8 + 4 = 21$ (variable part).

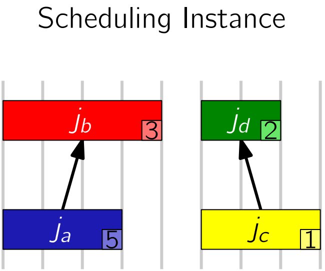
The algorithmic framework - Graph Structure

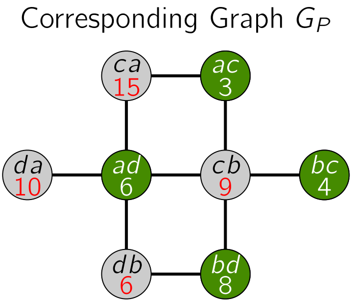




- Observe that any linear extension L defines a vertex cover of the graph. [$L = (a, b, c, d)$ defines **this vertex cover**.]

The algorithmic framework - Graph Structure





- Observe that any linear extension L defines a vertex cover of the graph. [$L = (a, b, c, d)$ defines **this vertex cover**.]
- Pairs that are reversed in L form an independent set in G_P.

Posets & (Hyper)graph of incomparable pairs

In Dimension theory the following Hypergraph $H_{\mathbf{P}}$ is well-known:

Vertices: (Ordered) Incomparable pairs

Hyperedges: (Minimal) Subsets of incomparable pairs that no linear extension can reverse simultaneously.

Dimension & Coloring

$\chi(H_{\mathbf{P}}) = \dim(\mathbf{P})$ [FELSNER & TROTTER'00]

$\chi_f(H_{\mathbf{P}}) = \mathrm{fdim}(\mathbf{P})$ [BRIGHTWELL & SCHEINERMANN'92]

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[Ambühl et al.'06] observed that the underlying graph (hyperedges of cardinality two) and the graph of [Correa & Schulz'04] **coincide**.

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The Approximability of the Fractional Dimension

Corrolary

If we have a "small" realizer, we have a "good" coloring of the Hypergraph $\Rightarrow$ good approximation for $1|\text{prec}|\sum_j w_j C_j$.

Theorem [Jain & Hegde'06]

It is hard to approximate the (fractional) dimension of a poset with n elements within a factor $n^{0.5-\varepsilon}$ for any $\varepsilon > 0$.

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However for several interesting posets we can do better...

Scheduling & Coloring - Applications

Using coloring:

Prec. Constr.	Other approaches	This approach
2-dimensional	$3/2$ [Correa & Schulz'04]	1
semi-orders	≈ 1.618 [Woeginger'03]	$4/3$
convex bipartite	≈ 1.618 [Woeginger'03]	$4/3$
interval-orders	≈ 1.618 [Woeginger'03]	2
interval dimension 2	2	2
Bounded degree d	2	2

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For remaining classes, use

- fractional coloring
- randomization

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Scheduling & Coloring - Applications

Using fractional coloring:

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convex bipartite	≈ 1.618 [Woeginger'03]	$4/3$
interval-orders	≈ 1.618 [Woeginger'03]	1.5
interval dimension 2	2	1.75
Bounded degree d	2	$2 - \frac{2}{d+1}$

For remaining classes, use

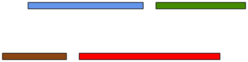
- fractional coloring
- randomization

Example - Interval Orders

- Interval orders is a well studied class of posets [Fishburn'85]
- [Woeginger'03] showed that $1|\text{prec}|\sum_j w_j C_j$ with interval order precedence constraints has a (≈ 1.61803)-approximation.

Interval Orders

A poset is an interval order if it can be represented by intervals such that $(a, b) \in P$ iff a's interval is completely before b's.



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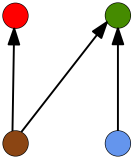
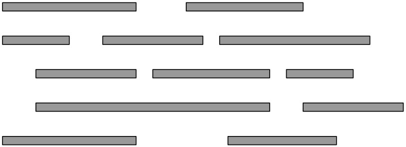


Interval orders can be recognized in $O(n^2)$

1.5-Approximation for Interval Orders



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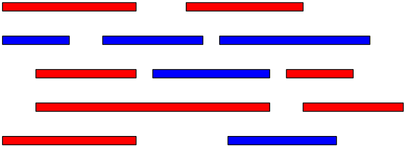
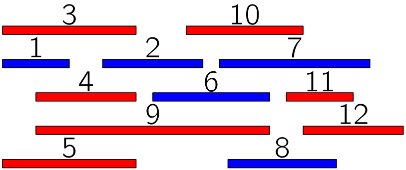


Partition the set of jobs into 2 sets (blue and red)

There are 2^n partitions

1.5-Approximation for Interval Orders



Partition the set of jobs into 2 sets (blue and red)

There are 2^n partitions

Lemma [RABINOVITCH'78]

For any (blue,red)-partition there is an L where blue jobs are scheduled before red jobs (if incomparable).

- Consider these $t = 2^n$ linear extensions
- Observe that there are $k = 2^{n-2}$ linear extensions where any inc. pair (a, b) is reversed: (a, b): **yes**, (a, b): **no**, (a, b): **maybe**, (a, b): **maybe**.
- This set of $t = 2^n$ linear extensions is a $k{:}t$-realizer ($t/k = 4$)

$$\alpha = \left(2 - \frac{2}{t/k}\right) = 2 - \frac{2}{4} = 1.5$$

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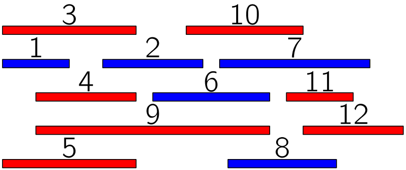

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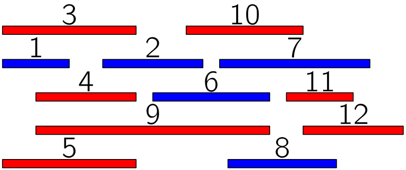

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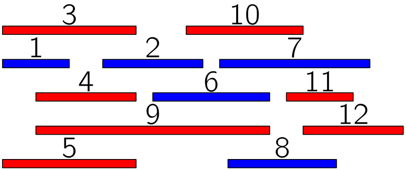

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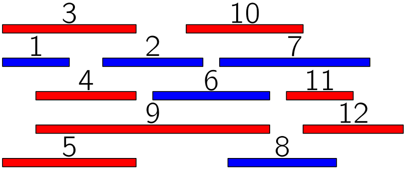

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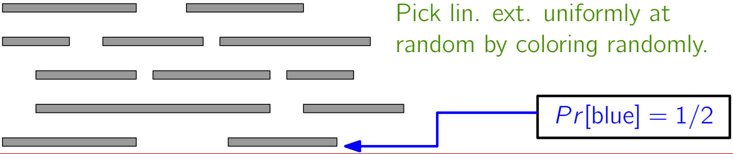
Problem

k:t-Realizer is of exponential size (2^n).

1.5-Approximation for Interval Orders

Pick lin. ext. uniformly at random by coloring randomly.



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Randomization: we only need to **sample** a good extension efficiently.

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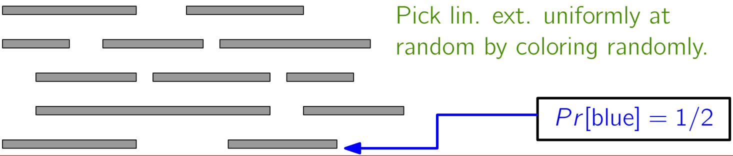

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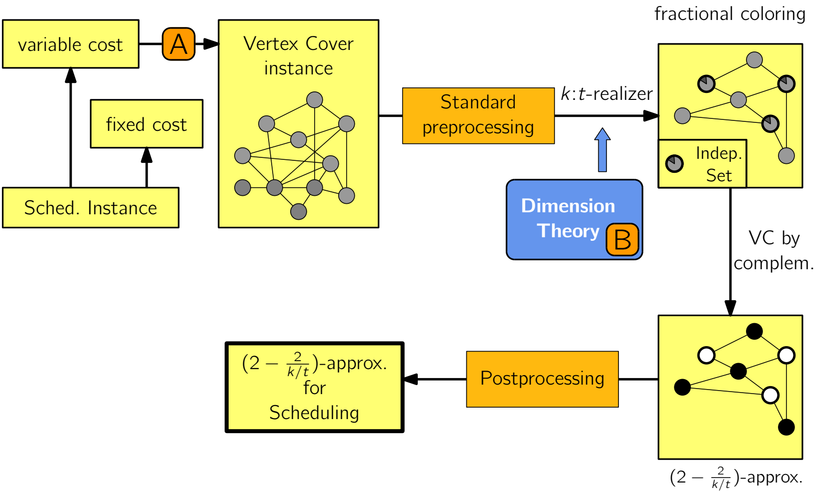
Note: Randomization is merely a "detour"

Method of Conditional Probabilities

$\Rightarrow$ deterministic 1.5-approximation algorithm.

The algorithmic Framework - Applications



The algorithmic Framework - Applications

Prec. Constr.	Other approaches	This approach
2-dimensional	$3/2$ [Correa & Schulz'04]	1
semi-orders	≈ 1.618 [Woeginger'03]	$4/3$
convex bipartite	≈ 1.618 [Woeginger'03]	$4/3$
interval-orders	≈ 1.618 [Woeginger'03]	1.5
interval dimension 2	2	1.75
Bounded degree d	2	$2 - \frac{2}{d+1}$

Summary

- Scheduling problem $\rightarrow$ weighted vertex cover on $G_\mathbf{P}$ ($+$ a fixed cost)
  - Adds more structure to the problem
  - 2-approximates the variable part!
  - Suggests a unified way of constructing the currently best-known approximation ratios for all considered posets.
- We did not improve the approximation ratio for the general case
  - No better than 2-approximation, assuming variant of UGC

    [BANSAL & KHOT'09]
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Outlook

- Hypergraph of incomparable pairs has nice properties:
  - Vertex Cover $\hat{=}$ **Feedback Arc Set**
  - Independent Set $\hat{=}$ **Maximum Acyclic Subgraph**
- $1|\text{prec}|\sum_j w_j C_j$ can be seen as Feedback Arc Set with "specially structured" weights.
- "Special structure" $\Rightarrow$ all hyperedges > 2 can be ignored!
- **Hypothesis**: *Other ordering problems lie in between.*
- Work in Progress: Rank Aggregation (with triangle inequality) $\hat{=}$ ignore hyperedges > 3

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Thank you for your attention!