Enumerating classes of regular triangulations

Vissarion Fisikopoulos
joint work with Ioannis Z. Emiris and Christos Konaxis

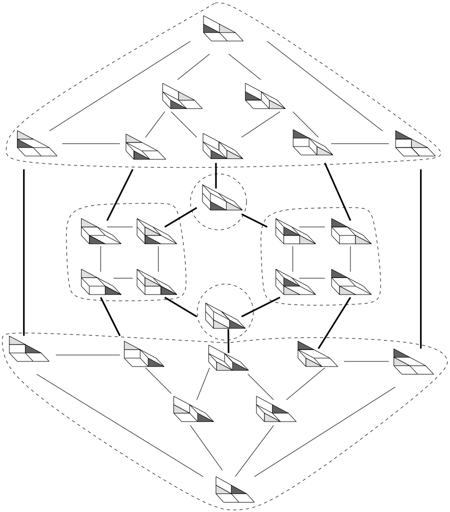
National and Kapodistrian University of Athens
Department of Informatics and Telecommunications

27 August 2010

Motivation

- **Computation of Resultants**
  - solve polynomial systems
- **Implicitization**
  - parametric (hyper)surfaces

- Reduction to graph enumeration problems



Outline

1. Triangulations and mixed subdivisions
   - definitions and the connection between them

2. Mixed cell configurations and R-equivalent classes
   - define equivalence classes of mixed subdivisions
   - flips between classes of mixed subdivisions

Outline

1. Triangulations and mixed subdivisions
   - definitions and the connection between them

2. Mixed cell configurations and R-equivalent classes
   - define equivalence classes of mixed subdivisions
   - flips between classes of mixed subdivisions

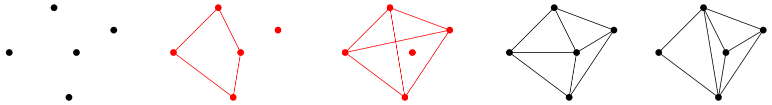
Triangulations

Definition

A **triangulation** of a point set A in $\mathbb{R}^d$ is a collection T of subsets of A called cells s.t.

- The cells cover $convex\_hull(A)$
- Every pair of cells intersect at a (possibly empty) common face
- Every cell is a simplex

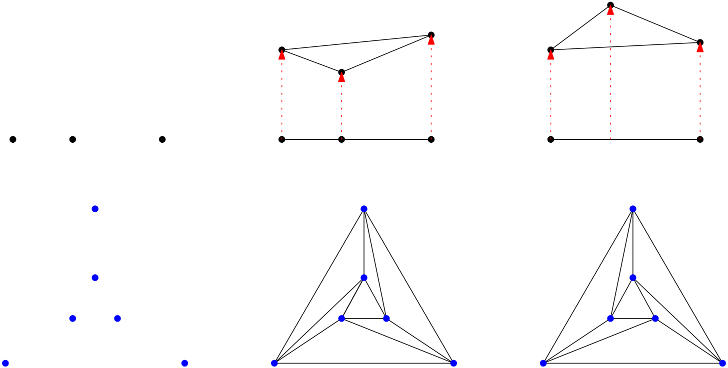
The operation of switching from one triangulation to another is called **flip**.



Regular Triangulations

Definition

A triangulation T of a point set A in $\mathbb{R}^d$ is **regular** if there exist a lifting function $w : A \to \mathbb{R}$ s.t. T is the projection to $\mathbb{R}^d$ of the lower hull of $\widetilde{A} = (a, w(a)), a \in A$.

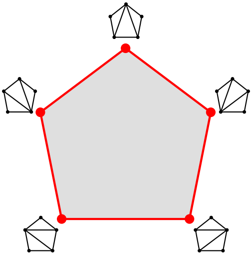


The Secondary Polytope

Let A a set of n points in $\mathbb{R}^d$.

Theorem [Gelfand-Kapranov-Zelevinsky]

To every point set A corresponds a Secondary polytope $\Sigma(A)$ with dimension $n - d - 1$. The vertices correspond to the regular triangulations of A and the edges to flips.



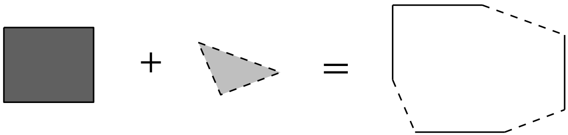
Enumeration of regular triangulations: [Rambau02], [Masada et al.96]

Minkowski Sum

Definition

The **Minkowski sum** of two convex polytopes P_1 and P_2 is the convex polytope:

$$P = P_1 + P_2 := \{p1 + p2 \mid p_1 \in P_1, p_2 \in P_2\}$$



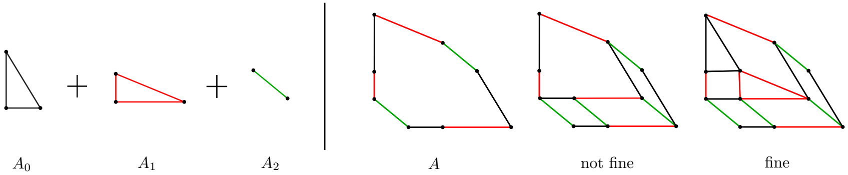
Mixed Subdivisions

Let $A_0, A_1, \ldots, A_d$ point sets in $\mathbb{R}^d$ and $A = A_0 + A_1 + \ldots + A_d$ their Minkowski sum.

Definition

A **fine mixed subdivision** of A is a collection of subsets (cells) of A s.t.

- the cells cover $convex\_hull(A)$ and intersect properly
- every cell $\sigma = F_0 + \cdots + F_d$ for $F_0 \subseteq A_0, \ldots, F_d \subseteq A_d$
- all F_i are affinely independent and σ does not contain any other cell



The Cayley Trick

Definition

The **Cayley embedding** of $A_0, \dots, A_d$ in $\mathbb{R}^d$ is the point set

$$\mathcal{C}(A_0, \dots, A_d) = A_0 \times \{e_0\} \cup \dots \cup A_d \times \{e_d\} \subseteq \mathbb{R}^d \times \mathbb{R}^d$$

where $e_0, \dots, e_d$ are an affine basis of $\mathbb{R}^d$.

Proposition (the Cayley trick)

$$\left(\begin{array}{c} \text{regular} \\ \text{triangulations} \\ \text{of} \\ \mathcal{C}(A_0, \dots, A_d) \end{array} \right) \longleftrightarrow \left(\begin{array}{c} \text{regular} \\ \text{fine mixed} \\ \text{subdivisions} \\ \text{of} \\ A_0 + \ldots + A_d \end{array} \right)$$

Outline

1. Triangulations and mixed subdivisions
   - definitions and the connection between them

2. Mixed cell configurations and R-equivalent classes
   - define equivalence classes of mixed subdivisions
   - flips between classes of mixed subdivisions

i-mixed cells

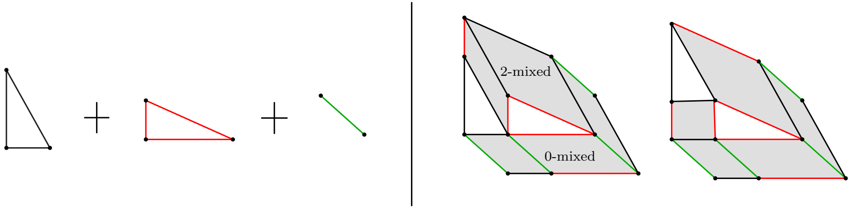
Let $A_0, A_1, \ldots, A_d$ and $A = A_0 + A_1 + \cdots + A_d$

Definition

A cell σ of a mixed subdivision is called **i-mixed** if for all j exists $F_j \subseteq A_j$ s.t.

$$\sigma = F_0 + \cdots + F_{i-1} + F_i + F_{i+1} + \cdots + F_d$$

where $|F_j| = 2$ (edges) for all $j \neq i$ and $|F_i| = 1$ (vertex).

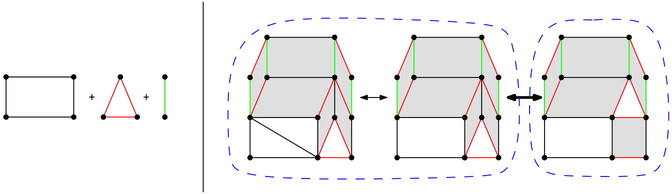


i - Mixed Cells Configurations

Generalizing mixed cells configurations of [MichielsVerschelde99]

Definition

i-mixed cells configurations are the equivalence classes of mixed subdivisions with the same i-mixed cells for all $i \in \{0, 1, \ldots, d\}$.

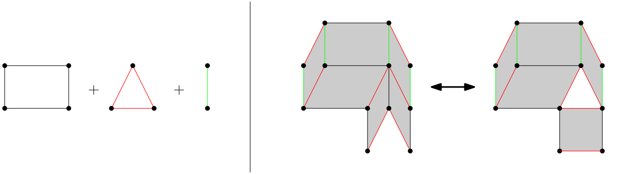


i - Mixed Cells Configurations

Generalizing mixed cells configurations of [MichielsVerschelde99]

Definition

i-mixed cells configurations are the equivalence classes of mixed subdivisions with the same i-mixed cells for all $i \in \{0, 1, \ldots, d\}$.

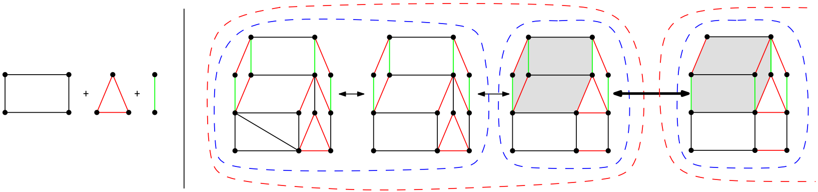


Proposition

There exist flips that transform one i-mixed cell configuration to another by destroying at least one i-mixed cell.

R-equivalent classes

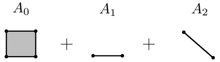
- The equivalence classes of mixed subdivisions as defined in [Sturmfels94].
- There exist flips, called **cubical** flips, that takes us from one R-equivalent class to another.

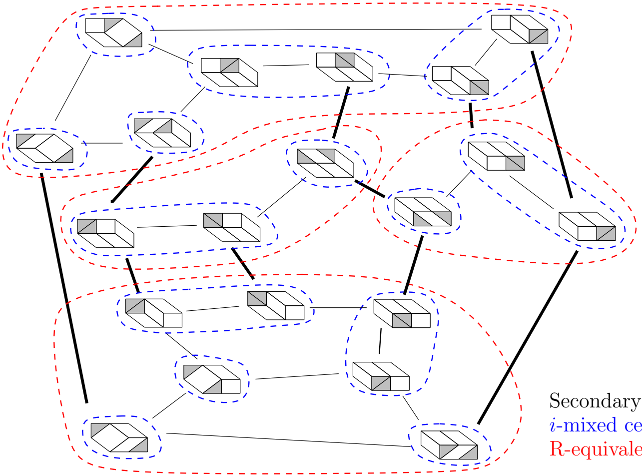


Proposition

A flip is cubical if and only if it involves exactly 2 points from each A_i.

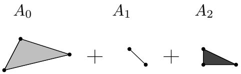
An illustration

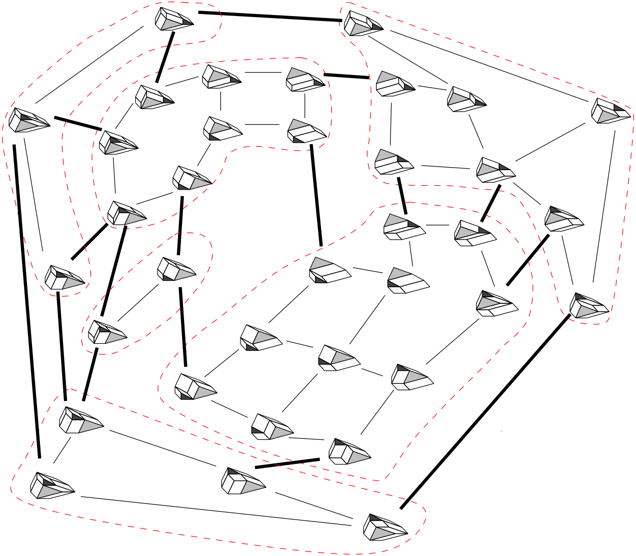




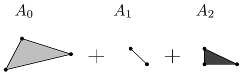
Secondary polytope
i-mixed cell configurations
R-equivalent classes

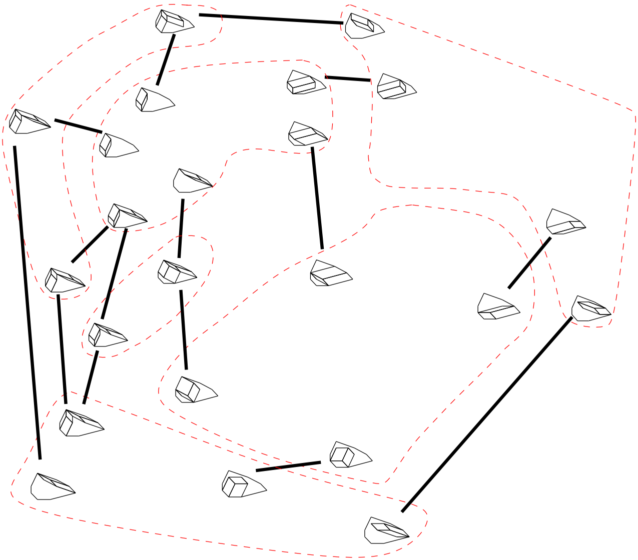
Disconnected graph of cubical flips



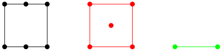
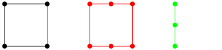
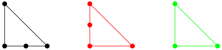


Disconnected graph of cubical flips



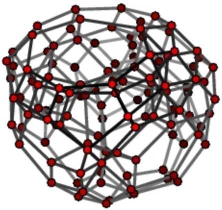


Complexity

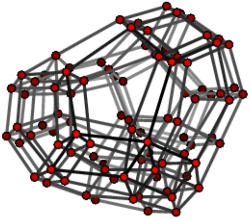
input point sets	# Secondary polytope vertices	i-mixed cell configurations	R-equivalent classes
	122	98	8
	104148	43018	21
	76280	32076	95
	3540	3126	22

i-mixed cell configurations

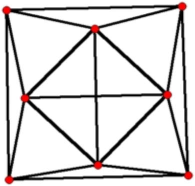
R-equivalence



Secondary Polytope



i-mixed cell configurations

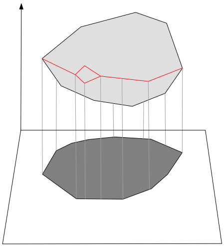


R-equivalent classes

Conclusion - Future Work

- # Σ vertices $\geq$ #i-mixed cell configurations $\geq$ #R-equivalent classes
- Algorithmic tests for flips between equivalent classes, disconnected graph of cubical flips
- Wiki page with experiments http://ergawiki.di.uoa.gr/index.php/Implicitization

- Enumerate R-equivalent classes
- The polytope defined by R-equivalent classes is a Minkowski summand of the Secondary polytope [MichielsCools00],[Sturmfels94]
- In some applications (e.g. implicitization) we need to compute only a silhouette w.r.t. a projection of this polytope [EmirisKonaxisPalios07]



Thank You!