Sponsored Search Auctions

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Introduction

Web search engines like Google and Yahoo! monetize their service by auctioning off advertising space next to their standard algorithmic search results.
For example, Apple or Best Buy may bid to appear among the advertisements – usually located above or to the right of the algorithmic results.
Introduction

- These sponsored results are displayed in a format similar to algorithmic results:
  - as a list of items each containing:
    - title,
    - text description
    - hyperlink to the advertiser’s Web page.
Introduction

We call each position in the list a *slot*.
free trips

Best Vegas Deals & Rates
56% - 75% Off Top Las Vegas Hotels. Lowest Rates & Best Deals in Vegas
www.TripRoo.com

Free Trip Sweepstakes
Enter To Win A Four-Star Celebrity Vacation! Visit & Learn More Today
www.Raisinets.com

Las Vegas Trips
Search and Compare Low Vegas Rates. Hotel Savings From 100s of Sites!
www.KAYAK.com

Vegas Trips
Deals to Las Vegas & More. Book a Package Now and Save w/ Orbitz!
www.ORBITZ.com

Free Trips Today Colorado
An extensive list of things to do in Denver that are completely free. ... They are located in downtown Denver and they have a lot of free days in 2011...
www.freetriptoday.com

Party Houses - Free Trips Out - big houses for hire UK, holiday...
Peak District Days Out for free, Party Houses – the places to go. ... Free Trips Out Planning Your Visit. Planning Your Visit. Extras... Spa Treatments. Entertainers ...
www.partyhouses.co.uk/page/151/Free_Trips_Out.htm

Taglit-Birthright Israel. Homepage
Taglit-Birthright Israel provides the gift of educational, first-time trips to Israel for Jewish young adults ages 18 to 26. Join us for the experience of a lifetime.
www.birthrightisrael.com/site/PageServer

Free Trips | Summit Group | Home | Business Info, by Jan & Randy Robinson

Tagged: Free Trips, Ireland, Summit Group, Summit Group Free Trip to New Zealand ... to enjoy our lives, and get great rewards from Watkins like these FREE trips ...
www.parttimeincome.com/tag/free-trips
Introduction

- More than 50% of Web users visit a search engine every day.
- Americans conduct roughly 6 billion Web searches per month.
- Over 13% of traffic to commercial sites is generated by search engines.
- Over 40% of product searches on the Web are initiated via search engines.
Today, Internet giants Google and Yahoo! boast a combined market capitalization of over $150 billion, largely on the strength of sponsored search.

Roughly 85% of Google’s $4.1 billion and roughly 45% of Yahoo!’s $3.7 billion in 2005 revenue is likely attributable to sponsored search.
Introduction

- Advertisers specify:
  - List of pairs of keywords
  - Bids
  - Total maximum daily or weekly budget.

- Every time a user searches for a keyword, an auction takes place among the set of interested advertisers who have not exhausted their budgets.
Existing Models

- **Static**
  - Vickrey Clarke Grooves Mechanism (VCG)
  - Generalized First Price (GFP)
  - Generalized Second Price (GSP)

- **Dynamic**
  - On-line Allocation Problem
Static

- $n$ bidders/advertisers
- $k$ slots ($k$ is fixed apriori – $k < n$)
- $\alpha_{ij}$ as a click through rate (CTR) of the bidder $j$ if placed in slot $i$
- $V_j$ is the value of the bidder $j$ for a click
Static

- Assumptions
  - Bidders prefer a higher slot to a lower slot
    \[ \alpha_{ij} \geq \alpha_{i+1,j} \text{ for } i=1,2,...,k-1 \]
  - \( V_i \) is independent of the slot position (static)
  - CTR for a slot does not depend on the identity of other bidders.
  - CTRs are assumed to be common knowledge (static nature)
  - not the reality - CTRs can fluctuate dramatically over small periods)
Static

- Revenue Maximization
- Allocative Efficiency
Revenue Maximization

- Result of Myerson
- The generalized Vickrey auction is applied not to the actual values $v_j$ but to the corresponding virtual values
- Generalized Vickrey auction with reserve prices
Revenue Maximization

Maximization bidder payments:

$$\max \sum_{j=1}^{n} p_j$$
Revenue Maximization

Surplus Allocation:
\[
\max \sum_{j=1}^{n} x_j(b)v_j
\]
x_j(b) : expected CTR of bidder j who bids b

Virtual Surplus Allocation:
\[
\max \sum_{j=1}^{n} x_j(b)\varphi_j(v_j)
\]
\[
\varphi_j(v_j) = v_j - \frac{1-F_j(v_j)}{f_j(v_j)}
\]

where:
\[
F_j(z) = \Pr[v_j \leq z], \quad f_j(z) = \frac{d}{dz} F_j(z)
\]

\(v_j\) : drawn ind/ntly from continuous prob. distribution
Revenue Maximization

- Expected **Profit** of a Truthful Mechanism $M$, is equal to the Expected Virtual Surplus:

$$E_t(M(t)) = E_t\left[ \sum_j \phi_j(v_j)x_j(t) \right]$$

**Proof:**

$$E_b(p_j(b)) = \int_{b=0}^{h} p_j(b)f(b)db = \ldots = E\left[ \phi_j(b)x_j(b) \right]$$

- **Mechanism Truthful in Expectation:**
  - $x_j(b)$ Monotone non-decreasing
  - $p_j(b) = b_jx_j(b) - \int_0^b x_j(z)dz$
Revenue Maximization

- Thus, Virtual surplus is truthful if and only if
  \[ \varphi_j(v_j) \] is monotone non-decreasing in \( v_j \)

- Myerson Mechanism:
  - Given bids \( b \) and \( F \) (here Bayesian – Nash distribution), compute ‘virtual bids’:
    \[ b'_i = \varphi_i(b_i) \]
  - Run VCG on \( b' \) to get \( x' \) and \( p' \)
  - Output \( x=x' \) and \( p \) with \( p_i = \varphi^{-1}_i(p'_i) \)
Revenue Maximization

- F is the Bayesian – Nash distribution of the generalized Vickrey (second price) auction (second price) with reserve prices.

- Proof similar with the Vickrey (second price) auction (second price) with reserve price for 1 item.
Revenue Maximization

- Revenue without reserve price:
  \[ R_0 = \frac{1}{3} \]

- Revenue with reserve price \( r \):
  \[ r = \frac{1}{2}, \quad R_{\frac{1}{2}} = \frac{5}{12} \]
Revenue Maximization

Revenue without reserve price:

- Given $V_A$, B’s valuation is likely to lie anywhere between 0 and $V_A$
- On average $V_B = V_A/2$
- On average, $V_B$ halfway between 0 and $V_A$
- On average, $V_A$ halfway between $V_B$ and 1
Revenue Maximization

- Revenue without reserve price:
  - \( E[V_B] = 1/3 \) and \( E[V_A] = 2/3 \)
  
  - \( E[V_B] = E[V_A]/2 = 1/3 \)
Revenue Maximization

- Revenue with reserve price $r$:
  - It may be the case that a bidder has positive valuation but negative virtual valuation.
  - Thus, for allocating a single item, the optimal mechanism finds the bidder with the largest nonnegative virtual valuation if there is one, and allocates to that bidder.
Revenue Maximization

- Revenue with reserve price \( r \):
  - bidder 1 (same for bidder 2) wins precisely when:
    
    \[
    \varphi_1(b_1) \geq \max\{\varphi_2(b_2), 0\} \quad \Rightarrow
    \]
    
    \[
    p_1 = \inf\{b : \varphi_1(b) \geq \varphi_2(b_2) \land \varphi_1(b) \geq 0\}
    \]
    
  - Since \( \varphi_1 = \varphi_2 = \varphi \)
    
    \[
    p_1 = \min\{b_1, \varphi^{-1}(0)\} = \varphi^{-1}(0)
    \]
    
  - For
    
    \[
    F(z) = z \quad , \quad f(z) = 1 \quad \Rightarrow \quad \varphi(z) = 2z - 1 \quad \Rightarrow \quad \varphi^{-1}(0) = \frac{1}{2}
    \]
Revenue Maximization

Revenue with reserve price r:

For r = 1/2:

- \( \Pr[\text{both below } 1/2] = 1/2 \times 1/2 = 1/4 \)
- \( \Pr[\text{both above } 1/2] = 1/2 \times 1/2 = 1/4 \)
- \( \Pr[\text{one above } 1/2] = 1/2 \)
- Est. payoff both below = 0
- Est. payoff both above = 4/6
- Est. payoff one above = 1/2

\[
R_{1/2} = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{4}{6} + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{12}
\]
Allocative Efficiency

- Let $x_{ij} = 1$ if bidder $j$ is assigned slot $i$

- $x_{ij} = 0$ otherwise
Solution of LP:

$$\max \sum_{i=1}^{k} \sum_{j=1}^{n} \alpha_{ij} v_j x_{ij}$$

s.t. $$\sum_{j=1}^{n} x_{ij} \leq 1 \quad \forall i=1,2,...,k$$

$$\sum_{i=1}^{k} x_{ij} \leq 1 \quad \forall j=1,2,...,n$$

$$x_{ij} \geq 0 \quad \forall i=1,2,...,k \quad \forall j=1,2,...,n$$
VCG

Dual:

\[ \min \sum_{i=1}^{k} p_i + \sum_{j=1}^{n} q_j \]

s.t. \[ p_i + q_j \geq \alpha_{ij} v_j \quad \forall i=1,2,\ldots,k \quad \forall j=1,2,\ldots,n \]

\[ p_i, q_j \geq 0 \quad \forall i=1,2,\ldots,k \quad \forall j=1,2,\ldots,n \]

\( p_i \): expected payment bidder

\( q_j \): expected profit bidder
Special Case:

- CTRs bidder independent:
  \[ \alpha_{ij} = \mu_i \]

- Simple algorithm Northwest Corner Rule:
  - Assign bidder with highest value top slot, second highest value second slot e.t.c
  - **Assortative** assignment
VCG

- Cons
  - requires solving a computational problem which needs to be done online for every search and is expensive
  - Other mechanisms better revenues than VCG
Let $b_1,\ldots,b_n$ be the bids. The GFP mechanism is as follows:

- Sorts bidders according to the bids $b_1,\ldots,b_n$.
- Assigns slots according to the order (assign top slot to the highest bidder and so on).
- Charge bidder $i$ according to his bid.

Yahoo! used a GFP auction until 2004.
Let $w_1, \ldots, w_n$ be the weights on bidders which are static and independent of the bids $b_1, \ldots, b_n$. The GSP mechanism is as follows:

- Sort bidders by $s_i = w_i b_i$
  - (assume $s_1 \geq s_2 \geq \ldots \geq s_n$)
- Allocate slots to bidders $1, \ldots, k$ in that order (i.e., bidder $i$ gets the $i$th slot if $i \leq k$).
- Charge $i$ the minimum bid he needs to retain his slot (i.e., $p_i = \frac{s_{i+1}}{w_i}$).
GSP

- Overture model: For every $i$, $w_i = 1$ (bidders ordered according to the bids only).

- Google model: Google assigns weights based on the CTR at the top slot $w_i \approx \alpha_{i1}$. The assumption here is that $\alpha_{i1}$ is static (or slow changing).

- This ordering is also called ‘revenue order’ since $s_i = \alpha_{i1} b_i$ is the expected revenue if $i$ is put in slot 1 and there is only one slot.
GFP not truthful

- Payoff in general: $c_{ij}(v_j - p_j)$

<table>
<thead>
<tr>
<th>Advertiser</th>
<th>$v_i$</th>
<th>$b_i$</th>
<th>Slot</th>
<th>$c_i$</th>
<th>$p_ic_i$</th>
<th>Total payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>50</td>
<td>40</td>
<td>1</td>
<td>10</td>
<td>400</td>
<td>100</td>
</tr>
<tr>
<td>Bob</td>
<td>20</td>
<td>19</td>
<td>2</td>
<td>5</td>
<td>95</td>
<td>5</td>
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<tr>
<td>Charlie</td>
<td>2</td>
<td>2</td>
<td>None</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
GSP not truthful

- Payoff in general: $c_{ij}(v_j - p_j)$

<table>
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<tr>
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<th>$c_i$</th>
<th>$p_i c_i$</th>
<th>Total payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>50</td>
<td>40</td>
<td>1</td>
<td>10</td>
<td>190</td>
<td>310</td>
</tr>
<tr>
<td>Bob</td>
<td>40</td>
<td>19</td>
<td>2</td>
<td>5</td>
<td>10</td>
<td>105</td>
</tr>
<tr>
<td>Charlie</td>
<td>2</td>
<td>2</td>
<td>None</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
GSP not truthful

- Payoff in general: $c_{ij}(v_j - p_j)$

Table 3: GSP example - true bids

<table>
<thead>
<tr>
<th>Advertiser</th>
<th>$v_i$</th>
<th>$b_i$</th>
<th>Slot</th>
<th>$c_i$</th>
<th>$p_i c_i$</th>
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</tr>
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<tr>
<td>Alice</td>
<td>50</td>
<td>50</td>
<td>1</td>
<td>10</td>
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</tr>
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<td>40</td>
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<td>5</td>
<td>10</td>
<td>190</td>
</tr>
<tr>
<td>Charlie</td>
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<td>2</td>
<td>None</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
GSP not truthful

- Payoff in general: $c_{ij}(v_j - p_j)$

<table>
<thead>
<tr>
<th>Advertiser</th>
<th>$v_i$</th>
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<th>Slot</th>
<th>$c_i$</th>
<th>$p_i c_i$</th>
<th>Total payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
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<td>3</td>
<td>2</td>
<td>5</td>
<td>10</td>
<td>240</td>
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<tr>
<td>Bob</td>
<td>40</td>
<td>40</td>
<td>1</td>
<td>10</td>
<td>30</td>
<td>370</td>
</tr>
<tr>
<td>Charlie</td>
<td>2</td>
<td>2</td>
<td>None</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
VCG Payoff

- Payoff in general:

\[ c_{ij}(v_j - p_j) \]

- Each bidder \( j \) would be made to pay the sum of

\[ (c_{i-1} - c_i)b_i \]

for every \( i \) below him

<table>
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<tr>
<td>Alice</td>
<td>50</td>
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<td>1</td>
<td>10</td>
<td>210</td>
<td>290</td>
</tr>
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<td>5</td>
<td>10</td>
<td>190</td>
</tr>
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<td>2</td>
<td>2</td>
<td>None</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
GSP vs VCG

- Search engines revenues under GSP better than VCG:

\[ c_i p_i^{\text{VCG}} - c_{i+1} p_{i+1}^{\text{VCG}} = (c_i - c_{i+1})b_{i+1} \leq c_i b_{i+1} - c_{i+1} b_{i+2} = c_i p_i - c_{i+1} p_{i+1} \]
Equilibrium Properties

- **GFP: Bayes-Nash symmetric equilibrium**
  - argument identical to that of the sealed bid **first price auction** for a single good for symmetric bidders (*same distributions*) the revenue equivalence theorem implies that revenue from GFP is the same as any other auction that allocates according to bid order.

- **Revenue Equivalence Principle** Under certain weak assumptions, for every two Bayesian–Nash implementations of the same social choice function $f$, we have that if for some type $t'$ of player $i$, the expected (over the types of the other players) payment of player $i$ is the same in the two mechanisms, then it is the same for every value of $i$’s type $t$. 
Equilibrium Properties

- GSP: Today nothing is known about the Bayesian equilibrium of the GSP auction
- Special Case:
  - CTRs are separable:
    \[ \alpha_{ij} = \mu_i \beta_j \]
  - special case:
    \[ \alpha_{ij} = \mu_i \]
- Locally Envy-Free equilibria
GSP Equilibrium Properties

- Retaliatiom:

  Suppose advertiser $k$ bids $b_k$ → assigned to position $i$, and advertiser $k'$ bids $b_{k'} > b_k$ → assigned to position $(i - 1)$.

  If $k$ raises his bid slightly, his own payoff does not change, but the payoff of the player above him decreases

  $k'$ can retaliate...
GSP Equilibrium Properties

- Vector of bids changes all time
- What if the vector converges to a rest point?
- An advertiser in position i should not want to "exchange" positions with the advertiser in position (i-1)
- "locally envy-free" vectors
An equilibrium of the simultaneous-move game ($\Gamma$) induced by GSP is locally envy-free if a player cannot improve his payoff by exchanging bids with the player ranked one position above him:

$$\mu_i v_{g(i)} - p_i \geq \mu_{i-1} v_{g(i)} - p_{i-1}$$
GSP Equilibrium Properties

LEMMA 1: The outcome of any locally envy-free equilibrium of auction $\Gamma$ is a stable assignment.

Proof:
- no advertiser can profitably rematch with a position assigned to an advertiser below him (equilibrium)

\[ \mu_i v_{g(i)} - p_i \geq \mu_{i+1} v_{g(i)} - p_{i+1} \]
Proof (cont):

- show that no advertiser can profitably rematch with the position assigned to an advertiser more than one spot above him.
- locally envyfree equilibrium: matching must be assortative

\[
\mu_i v_{g(i)} - p_i \geq \mu_{i+1} v_{g(i)} - p_{i+1}
\]

\[
\mu_{i+1} v_{g(i+1)} - p_{i+1} \geq \mu_i v_{g(i+1)} - p_i
\]

thus:

\[
(\mu_i - \mu_{i+1}) v_{g(i)} \geq (\mu_i - \mu_{i+1}) v_{g(i+1)}
\]
GSP Equilibrium Properties

Proof (cont):

Suppose $m \leq i$:

\[ \mu_i v_{g(i)} - p_i \geq \mu_{i-1} v_{g(i)} - p_{i-1} \]

\[ \mu_{i-1} v_{g(i-1)} - p_{i-1} \geq \mu_{i-2} v_{g(i-1)} - p_{i-2} \]

\[ \vdots \]

\[ \mu_{m+1} v_{g(m+1)} - p_{m+1} \geq \mu_m v_{g(m+1)} - p_m \]

Thus:

\[ \mu_i v_{g(i)} - p_i \geq \mu_m v_{g(i)} - p_m \]
LEMMA 2: If the number of advertisers is greater than the number of available positions then any stable assignment is an outcome of a locally envy-free equilibrium of auction $\Gamma$.

Proof:

- stable assignment $\Rightarrow$ assortative $\Rightarrow$ advertisers are labeled in decreasing order of their bids:

  $$ v_j > v_k \iff j < k $$

- Thus, advertiser $i$ match with position $i$, payment $i$
Proof (cont):

Let:

\[ b_1 = v_1 \]

and

\[ b_i = \frac{p_{i-1}}{\mu_{i-1}} \quad \text{for } i>1 \]
GSP Equilibrium Properties

**Proof (cont):**

- Let:

\[ b_i > b_{i+1} \]

otherwise:

\[
\frac{p_{i-1}}{\mu_{i-1}} \leq \frac{p_i}{\mu_i} \Rightarrow v_i - \frac{p_{i-1}}{\mu_{i-1}} \geq v_i - \frac{p_i}{\mu_i} \Rightarrow \mu_{i-1}v_i - p_{i-1} \geq \mu_i v_i - p_i
\]

- So, deviating and moving to a different position in this strategy profile is at most as profitable for any player as rematching with the corresponding position in the assignment game \( \Gamma \).
Let assign:

\[ p_i \rightarrow p_i^{VCG} \]

**THEOREM 1:** Strategy profile \( B^* \) is a locally envy-free equilibrium of game \( \Gamma \). In this equilibrium, each advertiser’s position and payment are equal to those in the dominant-strategy equilibrium of the game induced by VCG. In any other locally envy-free equilibrium of game \( \Gamma \), the total revenue of the seller is at least as high as in \( B^* \).
GSP Equilibrium Properties

Proof:
- Payments under strategy profile $B^*$ coincide with VCG $\Rightarrow B^*$ locally envy-free equilibrium (construction)
- This assignment is:
  - Best stable assignment for all advertisers
  - Worst stable assignment for auctioneers
GSP Equilibrium Properties

- In any stable assignment:

\[ p_k \geq \mu_{k+1} v_k = p_k^{\text{VCG}} \]

otherwise advertiser k+1 would find it profitable to match with position k. Next,

\[ p_{k-1} - p_k \geq (\mu_{k-1} - \mu_k) v_k \]

otherwise advertiser k would find it profitable to match with position k-1

\[ p_{k-1} - p_k \geq (\mu_{k-1} - \mu_k) v_k \implies p_{k-1} \geq (\mu_{k-1} - \mu_k) v_k + p_k = (\mu_{k-1} - \mu_k) v_k + p_k^{\text{VCG}} \geq p_{k-1}^{\text{VCG}} \]
Dynamic Aspects

- Online Allocation Problem
  - Auctions are repeated with great frequency
  - Model them as repeated games of incomplete information
  - For simplicity we assume that each page has only one slot for advertisements.
  - The objective is to maximize total revenue while respecting the budget constraint of the bidders
Online Allocation Problem

- n number of advertisers and m the number of keywords.
- advertiser j has a bid of $b_{ij}$ for keyword i and a total budget of $B_j$.
- Bids are small compared to budgets
- Since search engine has an accurate estimate of $r_i$, the number of people searching for keyword i for all $1 \leq i \leq m$, it is easy to approximate the optimal allocation using a simple LP
- $x_{ij}$ be the total number of queries on keyword i allocated to bidder j
Online Allocation Problem

LP:

\[
\text{max} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} x_{ij} \\
\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} \leq r_i \quad \forall 1 \leq i \leq m \\
\quad \sum_{i=1}^{m} b_{ij} x_{ij} \leq B_j \quad \forall 1 \leq j \leq n \\
\quad x_{ij} \geq 0 \quad \forall 1 \leq i \leq m, \quad \forall 1 \leq j \leq n
\]
Online Allocation Problem

Dual:

\[
\min \sum_{j=1}^{n} B_j \beta_j + \sum_{i=1}^{m} r_i \alpha_i \\
\text{s.t.} \quad \alpha_i + b_{ij} \beta_j \geq b_{ij} \quad \forall 1 \leq i \leq m, \forall 1 \leq j \leq n \\
\beta_j \geq 0 \quad \forall 1 \leq j \leq n \\
\alpha_i \geq 0 \quad \forall 1 \leq i \leq m
\]
Online Allocation Problem

- Complementary slackness:
  \[ b_{ij}(1-\beta_j) = \alpha' = \max b_{ik}(1-\beta_k), \ 1 \leq k \leq n \]

- Search engine allocates its corresponding advertisement space to the bidder \( j \) with the highest \( b_{ij} (1-\beta_j) \)

- if we allocate keyword \( i \) to agent now we obtain an immediate ‘payoff’ of \( b_{ij} \).

- However, this consumes \( b_{ij} \) of the budget ⇒ opportunity cost of \( b_{ij}\beta_j \).

- Reasonable to assign keyword \( i \) to \( j \) provided
  \[ b_{ij}(1-\beta_j) > 0 \]
Online Allocation Problem

- Greedy:
  - among the bidders whose budgets are not exhausted, allocate the query to the one with the highest bid

- competitive ratio—the ratio between online algorithm’s performance and the optimal offline algorithm's performance

- Competitive ratio of greedy algorithm is 1/2
Online Allocation Problem

- Greedy procedure is not guaranteed to find the optimum solution:
  - 2 bidders each with a budget of $2.
    - \( b_{11} = 2, b_{12} = 2 - \varepsilon, b_{21} = 2, b_{22} = \varepsilon \)
  - If query 1 arrives before query 2, it will be assigned to bidder 1.
  - bidder 1’s budget is exhausted. When query 2 arrives, it is assigned to bidder 2.
  - Objective Function value of \( 2 + \varepsilon \).
  - The optimal solution would assign query 2 to bidder 1 and query 1 to bidder 2, yielding an objective function value of \( 4 - \varepsilon \).
Online Allocation Problem

Similar to Graph Matching Problem:
- Consider the set G of girls matched in Mopt but not in Mgreedy
- Then every boy B adjacent to girls in G is already matched in Mgreedy: \(|B| \leq |M_{greedy}|\)
- There are at least \(|G|\) such boys (\(|G| \leq |B|\)) otherwise the optimal algorithm could, not have matched all the G girls. So: \(|G| \leq |M_{greedy}|\)
- By definition of G also:
  - \(|M_{opt}| \leq |M_{greedy}| + |G|\)
  - \(|M_{greedy}|/|M_{opt}| \geq 1/2\)
Online Allocation Problem

- Can we do better?
- BALANCE algorithm:
  - For each query, pick the advertiser with the largest unspent budget
Two advertisers A and B
A bids on query x, B bids on x and y
- Both have budgets of $4

Query stream: xxxxyyyyy
- BALANCE choice: ABABBBB__
- Optimal: AAAABBBBB

Competitive ratio = $\frac{3}{4}$
Analyzing BALANCE

Queries allocated to $A_1$ in optimal solution
Queries allocated to $A_2$ in optimal solution

Opt revenue = 2B
Balance revenue = 2B - x = B + y

We have $y \geq x$
Balance revenue is minimum for $x = y = B/2$
Minimum Balance revenue = $3B/2$
Competitive Ratio = $3/4$
BALANCE: General Result

- In the general case, worst competitive ratio of BALANCE is
  - $1 - 1/e \approx 0.63$

- Let’s see the worst case that gives this ratio
Worst Case for BALANCE

- N advertisers: A1, A2, ... AN
  - Each with budget B > N

- Queries: N·B queries appear in N rounds of B queries each:
  - Bidding: Round 1 queries: bidders A1, A2, ..., AN
  - Round 2 queries: bidders A2, A3, ..., AN
  - Round queries: bidders Ai, ..., AN

- Optimum allocation: Allocate round i queries to Ai
Worst Case for BALANCE
BALANCE Algorithm

- $\beta_j$’s as a function of the bidders spent budget
  \[ \phi(x) = 1 - e^{x-1} \]
  \[ \beta_j = 1 - \phi(f_j) \]

- $\beta_j$’s as a function of the bidders spent budget
- $f_j$: the fraction of the budget of bidder $j$, which has been spent

- **Algorithm**: Every time a query $i$ arrives, allocate its advertisement space to the bidder $j$, who maximizes $b_{ij}\phi(f_j)$
BALANCE Algorithm

- Let $k$ be a sufficiently large number used for discretizing the budgets of the bidders.
- Advertiser is of type $j$ if she has spent within $(j-1/k, j/k]$ fraction of budget so far.
- $s_j$: Total budget of type $j$ bidders.
- For $i = 0, 1, \ldots, k$, define $w_i$: Amount of money spent by all bidders from the interval $(i-1/k, i/k]$ of their budgets.
- Discrete version of function $\phi$:

$$\Phi(s) = 1 - \left(1 - \frac{1}{k}\right)^{k-s}$$
BALANCE Algorithm

- When $k$ tends to infinity:

\[ \Phi(s) \rightarrow \phi \left( \frac{s}{k} \right) \]

- Let $\text{OPT}$ be the solution of the optimal off-line algorithm
BALANCE Algorithm

- **Lemma:** At the end of the algorithm, this inequality holds:

\[
\sum_{i=0}^{k} \Phi(i)s_i \leq \sum_{i=0}^{k} \Phi(i)w_i
\]
BALANCE Algorithm

Lemma Proof:

- Consider time query $q$ arrives.
- OPT allocates $q$ to a bidder of current type $t$, whose type at the end of the algorithm will be $t'$. 
- $b_{opt}, b_{alg}$: amount of money that OPT and the BALANCE get from bidders for $q$.
- Let $i$ be the type of the bidder that the algorithm allocates the query.

$$\Phi(t')b_{opt} \leq \Phi(t)b_{opt} \leq \Phi(i)b_{alg}$$
Theorem: The competitive ratio of Algorithm 1 is $1 - \frac{1}{e}$.

Proof:

By definition:

$$w_i \leq \frac{1}{k} \sum_{j=i}^{k} s_j$$

Thus:

$$\sum_{i=0}^{k} \Phi(i)s_i \leq \frac{1}{k} \sum_{i=0}^{k} \Phi(i) \sum_{j=i}^{k} s_j$$

We conclude that:

$$\left( \Phi(0) - O\left( \frac{1}{k} \right) \right) \sum_{i=0}^{k} s_i \leq \sum_{i=0}^{k} \frac{i}{k} s_i$$

Note that as $k$ goes to infinity the left-hand side tends to $(1 - \frac{1}{e})$OPT. Right-hand revenue of the BALANCE
Bibliographic Notes

- Internet