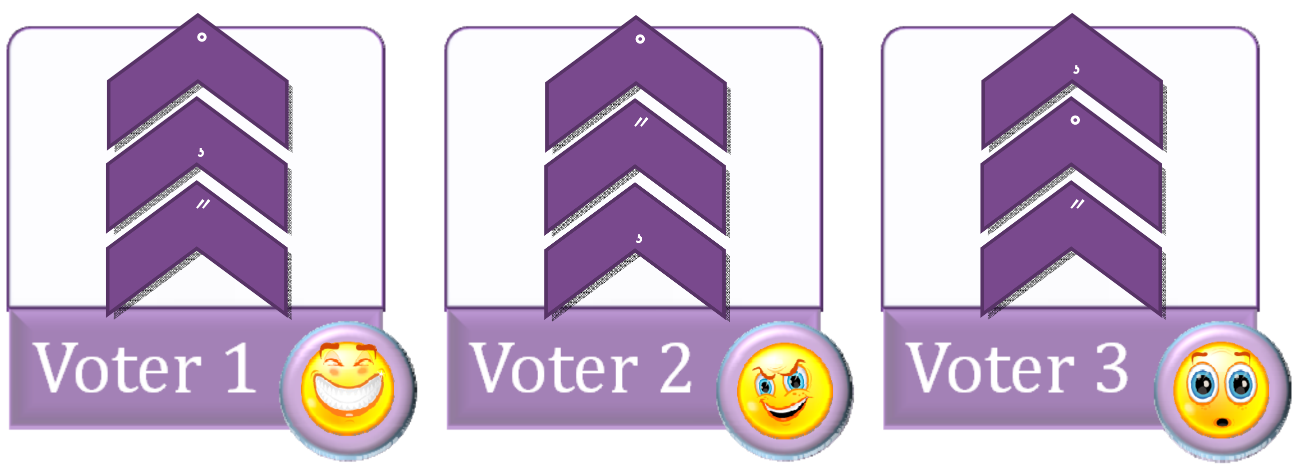
Socially desirable approximations for Dodgson's voting rule

Ioannis Caragiannis (University of Patras)
Christos Kaklamanis (University of Patras)
Nikos Karanikolas (University of Patras)
Ariel D. Procaccia (Harvard University)

Voting

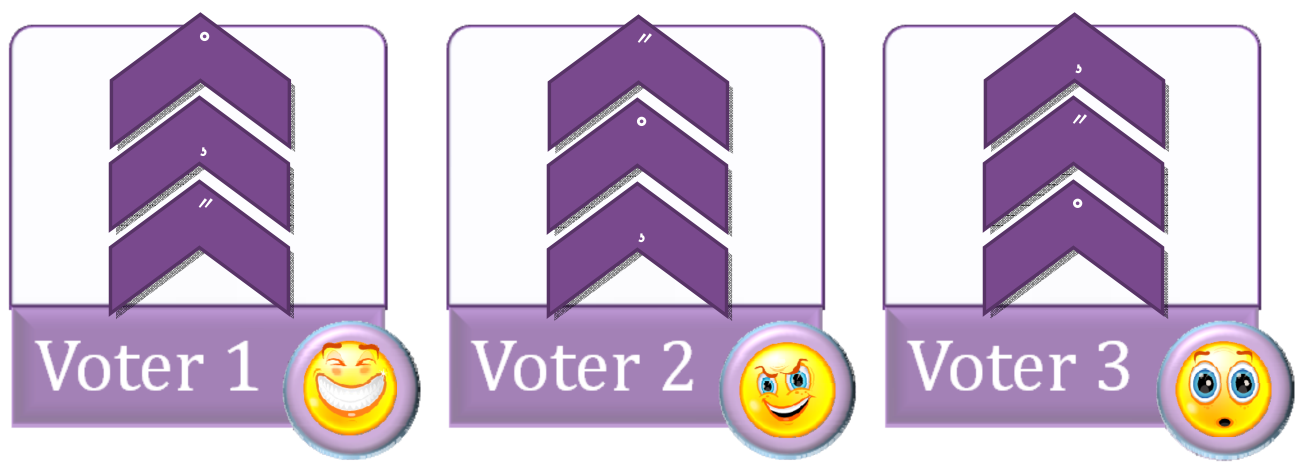
- n voters
- m candidates or alternatives
- n >>> m
- Voters rank the alternatives
- Preference profile: a vector of rankings



- Voting rule: a mapping of each preference profile to a winner, or a set of winners, or a ranking

Condorcet criterion

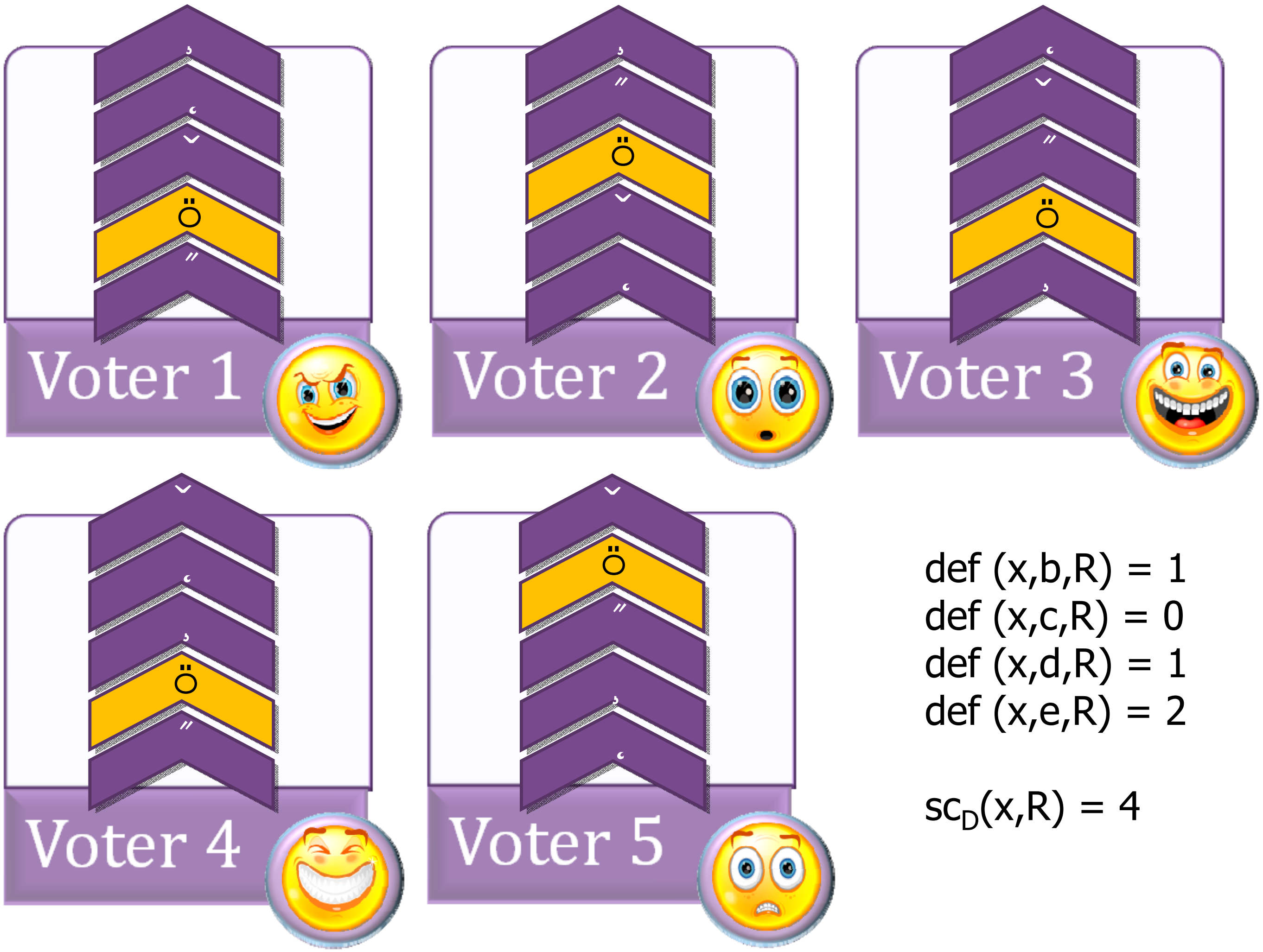
- Alternative x beats y in a pairwise election if the majority of voters prefers x to y
- Alternative x is a Condorcet winner if x beats any other alternative in a pairwise election
- Condorcet paradox: A Condorcet winner may not exist



Dodgson's voting rule

- Choose an alternative as close as possible to being a Condorcet winner according to some proximity measure
- Dodgson score of x: $sc_D(x,R)$
  - the minimum number of exchanges between adjacent alternatives needed to make x a Condorcet winner
  - alternatively: the total number of positions the voters push x

Dodgson's voting rule



def (x,b,R) = 1
def (x,c,R) = 0
def (x,d,R) = 1
def (x,e,R) = 2

$sc_D(x,R) = 4$

Dodgson's voting rule

- Choose an alternative as close as possible to being a Condorcet winner according to some proximity measure
- Dodgson score of x: $sc_D(x,R)$
  - the minimum number of exchanges between adjacent alternatives needed to make x a Condorcet winner
  - alternatively: the total number of positions the voters push x
- Dodgson ranking:
  - ranking of the alternatives in non-decreasing order of their Dodgson score
- Dodgson winner:
  - an alternative with the minimum Dodgson score

Related combinatorial problems

- Dodgson score (decision version):
  - Given a preference profile R, a particular alternative x, and an integer K, is the Dodgson score of x at most K? I.e., $sc_D(x,R) \leq K$?
- Dodgson score (optimization version):
  - Given a preference profile and a particular alternative x, what is the Dodgson score of x?
- Dodgson winner:
  - Given a preference profile and a particular alternative x, is x a Dodgson winner?
- Hard problems:
  - Bartholdi, Tovey, and Trick (Social Choice & Welfare, 1989)
  - Hemaspaandra, Hemaspaandra, and Rothe (J. ACM, 1997)

Approximation algorithms

- Approximation algorithms compute approximate Dodgson scores
- An algorithm V is a Dodgson approximation with approximation ratio ρ if given a preference profile R and a particular alternative x, computes a score $sc_V(x,R)$ for x such that
  - $sc_D(x,R) \leq sc_V(x,R) \leq \rho\, sc_D(x,R)$
- There exist polynomial-time Dodgson approximations with approx. ratio at most $H_{m-1} \leq 1+\ln m$
  - A greedy combinatorial algorithm
  - An algorithm based on linear programming
- Hard to approximate the Dodgson score within a factor better than $(1/2-\varepsilon)\ln m$
  - C., Covey, Feldman, Homan, Kaklamanis, Karanikolas, Procaccia, Rosenschein (SODA 09)

Approximation algorithms as voting rules

- Dodgson approximations are new voting rules
  - Simply rank the alternatives according to their score
- How good are they as voting rules?
  - Any Dodgson approximation with finite approx. ratio is Condorcet consistent
  - What about other social choice properties?

Compare to Dodgson

- The Dodgson rule satisfies
  - Condorcet consistency (by definition)
- but not
  - Monotonicity
  - Homogeneity
  - Combinativity
  - Smith consistency
  - Mutual majority consistency
  - Invariant loss consistency
  - Independence of clones
- Fishburn (SIDMA 77), Tideman (2006), Brandt (Math. Logic Q. 09)

The main question

- What is the best possible approx. ratio of Dodgson approximations that satisfy
  - Monotonicity?
  - Homogeneity?
  - Combinativity?
  - Smith consistency?
  - Mutual majority consistency?
  - Invariant loss consistency?
  - Independence of clones?
- In other words, how far is Dodgson’s voting rule from these properties?

Overview of results

<table>
<tr><th>Social Choice property</th><th>Approx. ratio lower bound</th><th>Approx. ratio upper bound</th><th>Time</th></tr>
<tr><td>Monotonicity</td><td>2
$(1/2-\varepsilon)\ln m$</td><td>2
$2H_{m-1}$</td><td>exp.
poly</td></tr>
<tr><td>Homogeneity</td><td>$\Omega(m \ln m)$</td><td>$O(m \ln m)$</td><td>poly</td></tr>
<tr><td>Combinativity
Smith consistency</td><td>$\Omega(nm)$</td><td colspan="2" rowspan="2">Trivial upper bound of $O(nm)$</td></tr>
<tr><td>Mutual majority consistency
Invariant loss consistency
Independence of clones</td><td>$\Omega(n)$</td></tr>
</table>

Monotonicity

- A voting rule is monotonic when for any profile R' and a profile R that is obtained from R' by pushing a single alternative x upwards in the preferences of some voters, the following holds:
  - If x is a winner in R', it is also a winner in R

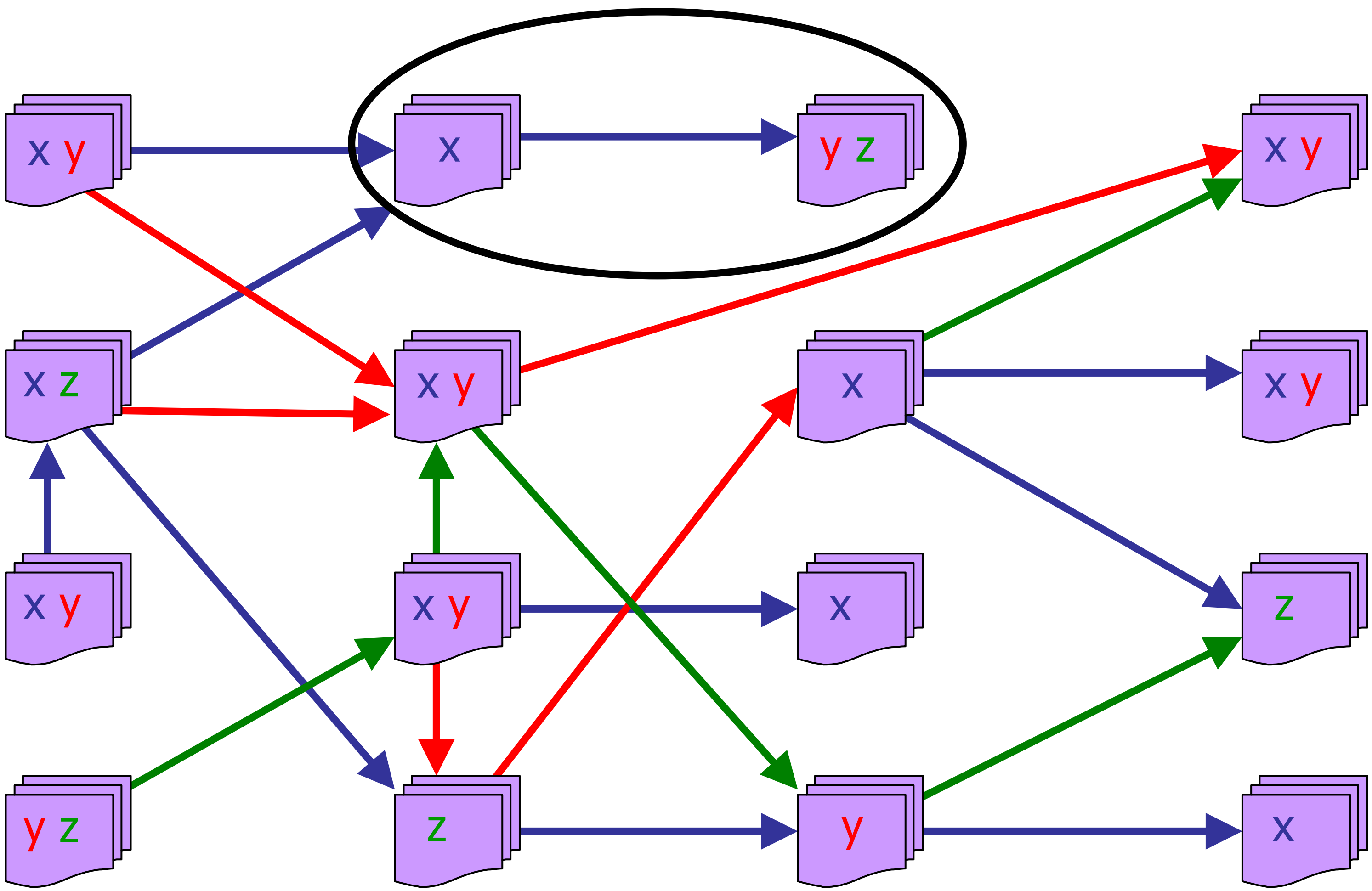
Monotonization

- What modifications a voting rule requires in order to become monotonic?
- E.g., for Dodgson:
  - Construct a new voting rule by considering all profiles
  - First decide which the winning set W(R) of alternatives for each profile R should be so that monotonicity is preserved

  - Then adjust the scores accordingly so that the resulting rule is a Dodgson approximation (with good approx. ratio, if possible)

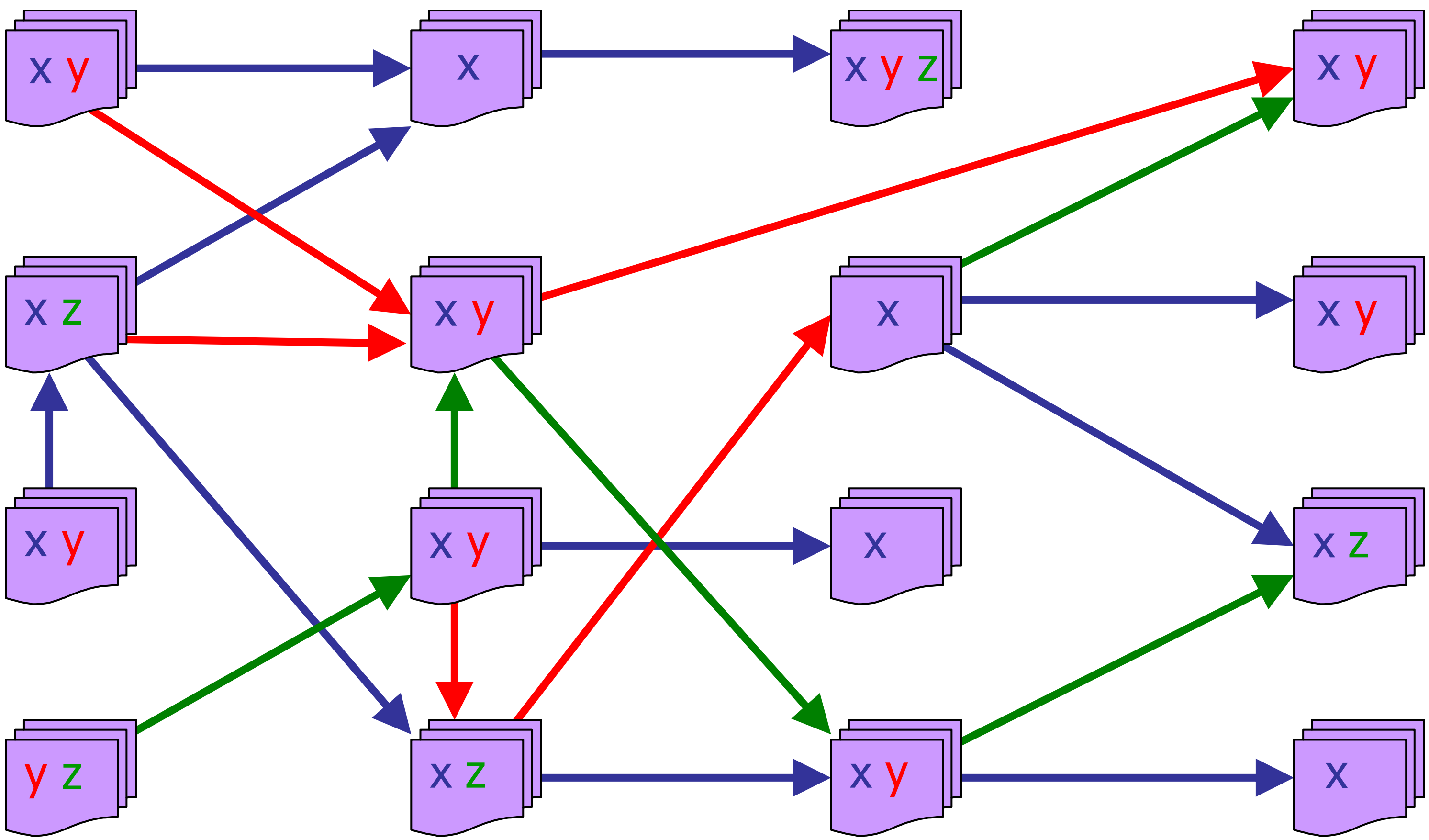
Monotonization

- A non-monotonic voting rule



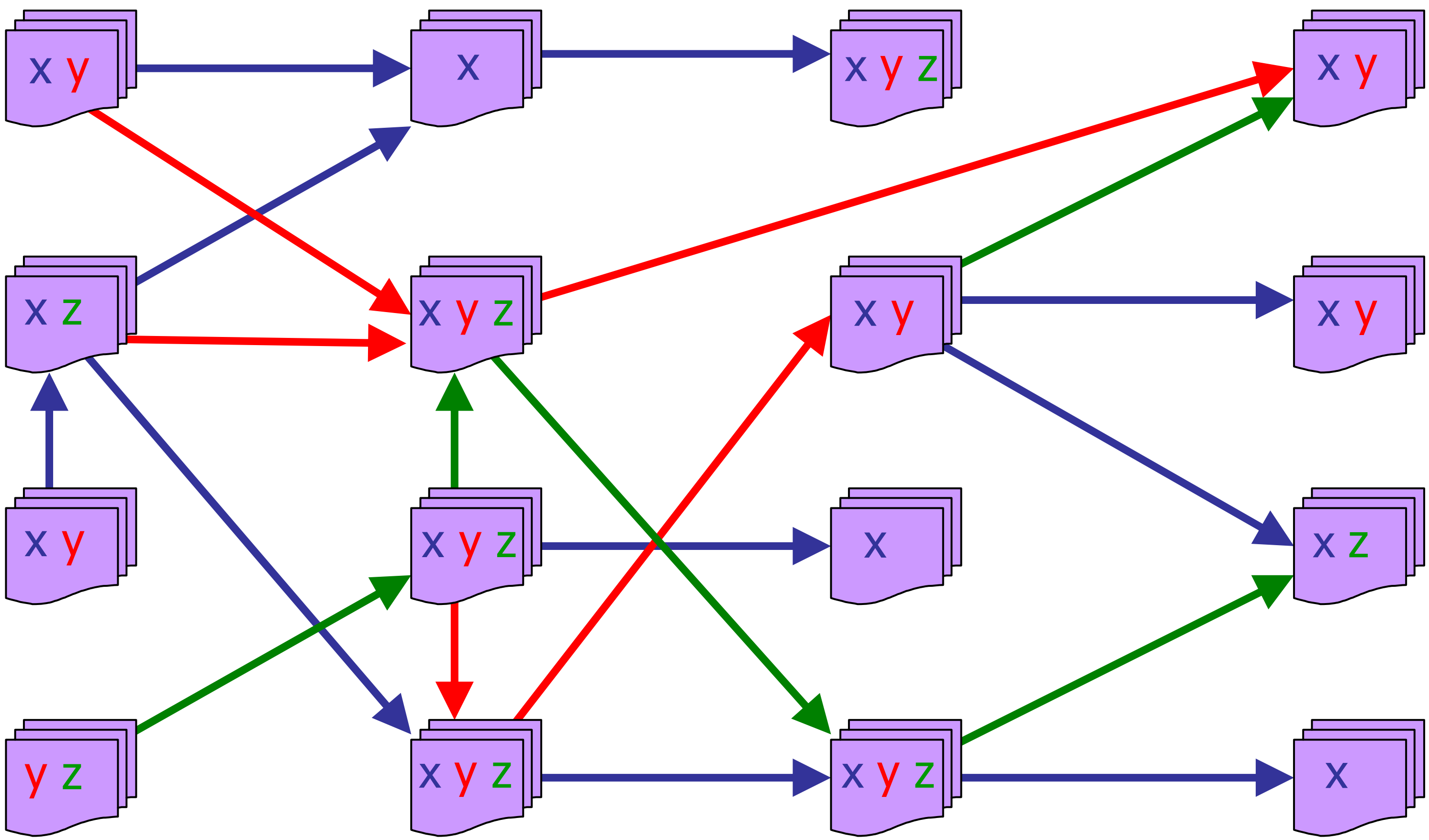
Monotonization

- Propagate x through the blue arcs, and similarly for y and z



Monotonization

- A monotonic voting rule M



Monotonization

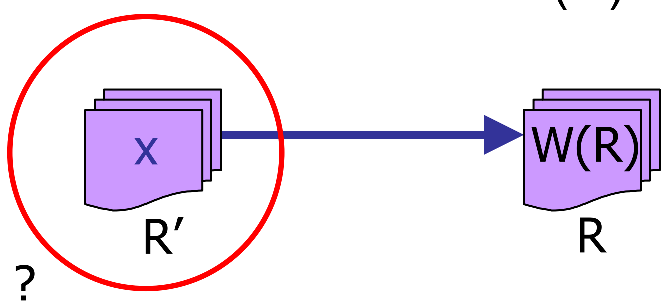
- Adjust the scores in order to obtain M:
  - Let Δ be the maximum Dodgson score of the alternatives in W(R)
  - Set $sc_M(x,R) = \Delta$ for each alternative in W(R)
  - Set $sc_M(y,R) = \max\{\Delta+1, sc_D(y,R)\}$ for any other alternative

Upper bounds for monotonic Dodgson approximations

- Monotonizing Dodgson yields a Dodgson approximation with approx. ratio 2
  - Intuition: pushing an alternative upwards can decrease the Dodgson score of another alternative up to half
  - Optimal approx. ratio
  - Polynomial-time if m is constant
  - Exponential-time in general
- Monotonizing the LP-based Dodgson approximation can be done in polynomial-time
  - Yields an approximation ratio of $2H_{m-1}$
  - Using a tool we call pessimistic estimator

Pessimistic estimators

- Given a profile R with winning alternatives W(R) according to the LP-based Dodgson approximation, and an alternative x not in W(R)
  - is there any profile R’ so that R is obtained from R’ by pushing x upwards in some voters
  - so that x wins some alternative in W(R) in R’?



- Our pessimistic estimators work in polynomial time by solving linear programs and are correct when answering NO
- Loss of an extra factor of 2 in the approx. ratio

Homogeneity

- A voting rule is homogeneous when for each profile R with a winning alternative x, x is also a winning alternative in any profile which is produced by replicating R

- Tideman (2006)
  - If there exists a Condorcet winner, then this is the winner
  - Otherwise, set

$$td(x,R) = \sum_{y \in A-\{x\}} \max\{0, losses(x,y,R\} - wins(x,y,R)\}$$

  - and rank the alternatives according to this score

- This rule is homogeneous and monotonic
- Is it a Dodgson approximation?
  - At first glance: No

Tideman's simplified Dodgson rule

- An alternative definition
  - If x is a Condorcet winner, then $sc_{td}(x,R) = 0$
  - Otherwise $sc_{td}(x,R) = m\ td(x,R) + m\log m$
- The alternative definition of Tideman's simplified voting rule yields a Dodgson approximation with approx. ratio $O(m\log m)$

Are there better homogeneous Dodgson approximations?

- No! Any homogeneous Dodgson approximation has approx. ratio $\Omega(m\log m)$
- Proof idea: Construction of a profile so that
  - An alternative x is tied against $\Omega(m)$ other alternatives and has Dodgson score $\Theta(m\log m)$
  - Another alternative y has deficit 2 against some alternative and Dodgson score 2
  - By duplicating the profile, the Dodgson score of x stays $\Theta(m\log m)$ but the Dodgson score of y pumps up
  - Still, due to homogeneity, the winner in the original profile should be a winner in the duplicated one

Social Choice and Computational Complexity

- Computational Complexity Theory provides the tools to understand computational aspects of voting rules
  - Negative results: Hardness of computation/approximation (e.g., Dodgson's voting rule)
  - Positive results: Approximation algorithms that could be used as alternative voting rules
- Besides statements about efficiency of computation, what other feedback can CCT give to SCT?
  - Are there approximation algorithms for a given voting rule that can be used as alternative voting rules with desirable social choice properties?
  - How far from a desirable social choice property is a given voting rule?

Open problems

- What about approximations of other voting rules?
- Different notions of approximation (additive, differential, approximation of rankings, etc.)
- Approximability of a voting rule by known rules that have good social choice properties (e.g., Copeland, Maximin)